Lecture Notes in Mathematics 1897

Subseries:
Ecole d'Eté de Probabilités de Saint-Flour

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Fluctuation Theory for Lévy Processes

Ecole d'Eté de Probabilités de Saint-Flour XXXV - 2005

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Springer

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Cover: Blaise Pascal (1623-1662)

Library of Congress Control Number: 2007921692

Mathematics Subject Classification (2000): 60G51, 60G10, 60G17, 60J55, 60J75

ISSN print edition: 0075-8434
ISSN electronic edition: 1617-9692
ISSN Ecole d'Eté de Probabilités de St. Flour, print edition: 0721-5363
ISBN-10 3-540-48510-4 Springer Berlin Heidelberg New York
ISBN-13 978-3-540-48510-0 Springer Berlin Heidelberg New York
DOI 10.1007/978-3-540-48511-7

Springer is a part of Springer Science+Business Media
springer.com

Typesetting: by the author and SPi using a Springer LaTeX package

Cover design: *design & production* GmbH, Heidelberg

Printed on acid-free paper SPIN: 11917915 VA41/3100/SPi 5 4 3 2 1 0

Foreword

The Saint-Flour Probability Summer School was founded in 1971. It is supported by CNRS, the "Ministère de la Recherche", and the "Université Blaise Pascal".

Three series of lectures were given at the 35th School (July 6–23, 2005) by the Professors Doney, Evans and Villani. These courses will be published separately, and this volume contains the course of Professor Doney. We cordially thank the author for the stimulating lectures he gave at the school, and for the redaction of these notes.

53 participants have attended this school. 36 of them have given a short lecture. The lists of participants and of short lectures are enclosed at the end of the volume.

Here are the references of Springer volumes which have been published prior to this one. All numbers refer to the *Lecture Notes in Mathematics* series, except S-50 which refers to volume 50 of the *Lecture Notes in Statistics* series.

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Jean Picard
Clermont-Ferrand, April 2006

Contents

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Introduction to Lévy Processes

Lévy processes, i.e. processes in continuous time with stationary and independent increments, are named after Paul Lévy: he made the connection with infinitely divisible distributions (Lévy–Khintchine formula) and described their structure (Lévy–Itô decomposition).

I believe that their study is of particular interest today for the following reasons

- They form a subclass of general Markov processes which is large enough to include many familiar processes such as Brownian motion, the Poisson process, Stable processes, etc, but small enough that a particular member can be specified by a few quantities (the *characteristics* of a Lévy process).
- In a sense, they stand in the same relation to Brownian motion as general random walks do to the simple symmetric random walk, and their study draws on techniques from both these areas.
- Their *sample path behaviour* poses a variety of difficult and fascinating questions, some of which are not relevant for Brownian motion.
- They form a flexible class of models, which have been applied to the study of storage processes, insurance risk, queues, turbulence, laser cooling, ... and of course finance, where the feature that they include examples having "heavy tails" is particularly important.

This course will cover only a part of the theory of Lévy processes, and will not discuss applications. Even within the area of fluctuation theory, there are many recent interesting developments that I won't have time to discuss.

Almost all the material in Chapters 1–4 can be found in Bertoin [12]. For related background material, see Bingham [19], Satô [90], and Satô [91].

1.1 Notation

We will use the canonical notation, and denote by $X = (X_t, t \geq 0)$ the co-ordinate process, i.e. $X_t = X_t(\omega) = \omega(t)$, where $\omega \in \Omega$, the space of real-valued cadlag paths, augmented by a cemetery point ϑ, and endowed with the Skorohod topology. The Borel σ-field of Ω will be denoted by $\mathcal{F}$ and the lifetime by $\zeta = \zeta(\omega) = \inf\{t \geq 0 : \omega(t) = \vartheta\}$.

Definition 1. *Let $\mathbb{P}$ be a probability measure on $(\Omega, \mathcal{F})$ with $\mathbb{P}(\zeta = \infty) = 1$. We say that X is a (real-valued) Lévy process for $(\Omega, \mathcal{F}, \mathbb{P})$ if for every $t \geq s \geq 0$, the increment $X_{t+s} - X_t$ is independent of $(X_u, 0 \leq u \leq t)$ and has the same distribution as X_s.*

Note that this forces $\mathbb{P}(X_0 = 0) = 1$; we will later write $\mathbb{P}_x$ for the measure corresponding to $(x + X_t, t \geq 0)$ under $\mathbb{P}$.

(Incidentally the name Lévy process has only been the accepted terminology for approximately 20 years; prior to that the name "process with stationary and independent increments" was generally used.)

From the decomposition

$$X_1 = X_{\frac{1}{n}} + \left(X_{\frac{2}{n}} - X_{\frac{1}{n}}\right) + \cdots + \left(X_{\frac{n}{n}} - X_{\frac{n-1}{n}}\right)$$

it is apparent that X_1 has an *infinitely divisible* distribution under $\mathbb{P}$. The form of a general infinitely divisible distribution is given by the well-known Lévy–Khintchine formula, and from it we deduce easily the following result.

Theorem 1. *Let X be a Lévy process on $(\Omega, \mathcal{F}, \mathbb{P})$; then*

$$\mathbb{E}(\exp i\lambda X_t) = e^{-t\Psi(\lambda)}, \ t \geq 0, \lambda \in \mathbb{R},$$

where, for some real γ, σ and measure Π on $\mathbb{R} - \{0\}$ which satisfies

$$\int_{-\infty}^{\infty} \{x^2 \wedge 1\} \Pi(dx) < \infty, \tag{1.1.1}$$

$$\Psi(\lambda) = -i\gamma\lambda + \frac{\sigma^2}{2}\lambda^2 + \int_{-\infty}^{\infty} \left\{1 - e^{i\lambda x} + i\lambda x \mathbf{1}_{(|x|<1)}\right\} \Pi(dx). \tag{1.1.2}$$

Ψ is called the **Lévy exponent** of X, and we will call the quantities γ the linear cefficient, σ the Brownian coefficient, and Π the Lévy measure of X : together they constitute the **characteristics** of X. There is an existence theorem: given real γ, any $\sigma \geq 0$ and measure Π satisfying (1.1.1) there is a measure under which X is a Lévy process with characteristics γ, σ and Π. There is also a uniqueness result, as any alteration in one or more of the characteristics results in a Lévy process with a different distribution.

Examples

- The characteristics of standard Brownian motion are $\gamma = 0, \sigma = 1, \Pi \equiv 0$, and $\Psi(\lambda) = \frac{\lambda^2}{2}$.
- The characteristics of a compound Poisson process with jump rate c and step distribution F are

$$\gamma = c \int_{\{|x|<1\}} xF(dx), \sigma = 0, \Pi(dx) = cF(dx),$$

  and $\Psi(\lambda) = c(1 - \phi(\lambda))$, where $\phi(\theta) = \int_{-\infty}^{\infty} e^{i\lambda x} dF(x)$.

- The characteristics of a Gamma process are

$$\gamma = c(1 - e^{-1}), \sigma = 0, \Pi(dx) = cx^{-1}e^{-x}\mathbf{1}_{\{x>0\}}dx,$$

and $\Psi(\lambda) = c \log(1 - i\lambda)$.
- The characteristics of a strictly stable process of index $\alpha \in (0,1)\cup(1,2)$ are

$$\gamma \text{ arbitrary}, \sigma = 0, \Pi(dx) = \begin{matrix} c_+ x^{-\alpha-1}dx & \text{if } x > 0, \\ c_-|x|^{-\alpha-1}dx & \text{if } x < 0. \end{matrix}$$

If $\alpha \neq 1$, $c_+ \geq 0$ and $c_- \geq 0$ are arbitrary, and

$$\Psi(\lambda) = c|\lambda|^\alpha \{1 - i\beta sgn(\lambda)\tan(\pi\alpha/2)\} - i\gamma\lambda.$$

If $\alpha = 1$, $c_+ = c_- > 0$, and $\Psi(\lambda) = c|\lambda| - i\gamma\lambda$; this is a Cauchy process with drift.

Note that there is a fairly obvious generalisation of Theorem 1 to $\mathbb{R}^d$, but we will stick, almost exclusively, to the 1-dimensional case.

The first step to getting a probabilistic interpretation of Theorem 1 is to realise that the process of jumps,

$$\Delta = (\Delta_t, t \geq 0) \text{ where } \Delta_t = X_t - X_{t-},$$

is a Poisson point process, but first we need some background material.

1.2 Poisson Point Processes

A random measure ϕ on a Polish space E (this means it is metric-complete and separable) is called a Poisson measure with intensity ν if

1. ν is a σ-finite measure on E;
2. for every Borel subset B of E with $0 < \nu(B) < \infty$, $\phi(B)$ has a Poisson distribution with parameter $\nu(B)$; in particular $\phi(B)$ has mean $\nu(B)$;
3. for disjoint Borel subsets $B_1, \cdots B_n$ of E, the random variables $\phi(B_1), \cdots, \phi(B_n)$ are independent.

In the case that $c := \nu(E) < \infty$, it is clear that we can represent ϕ as a sum of Dirac point masses as follows. Let $y_1, y_2, \cdots$ be a sequence of independent and identically distributed E-valued random variables with distribution $c^{-1}\nu$, and N an independent Poisson-distributed random variable with parameter c; then we can represent ϕ as

$$\phi = \sum_1^N \delta_{y_j},$$

where δ_y denotes the Dirac point mass at $y \in E$. If $\nu(E) = \infty$, there is a decomposition of E into disjoint Borel sets $E_1, E_2, \cdots$, each having $\nu(E_j)$

finite, and we can represent ϕ as the sum of independent Poisson measures ϕ_j having intensities $\nu\mathbf{1}_{E_j}$, each having the above representation, so again ϕ can be represented as the sum of Dirac point masses.

To set up a Poisson point process we consider the product space $E\times[0,\infty)$, the measure $\mu=\nu\times dx$, and a Poisson measure ϕ on $E\times[0,\infty)$ with intensity μ. It is easy to check that a.s. $\phi(E\times\{t\})=1$ or 0 for all $t\geq 0$, so we can introduce a process $(e(t),t\geq 0)$ by letting $(e(t),t)$ denote the position of the point mass on $E\times\{t\}$ in the first case, and in the second case put $e(t)=\xi$, where ξ is an additional isolated point. Then we can write

$$\phi=\sum_{t\geq 0}\delta_{(e(t),t)}.$$

The process $e=(e(t),t\geq 0)$ is called a Poisson point process with characteristic measure ν.

The basic properties of a Poisson point process are stated in the next result.

Proposition 1. *Let B be a Borel set with $\nu(B)<\infty$, and define its counting process by*

$$N_t^B=\#\{s\leq t: e(s)\in B\}=\phi(B\times[0,t]),\ t\geq 0,$$

and its entrance time by

$$T_B=\inf\{t\geq 0: e(t)\in B\}.$$

Then

(i) N^B is a Poisson process of parameter $\nu(B)$, which is adapted to the filtration $\mathcal{G}$ of e.
(ii) T_B is a $(\mathcal{G}_t)$-stopping time which has an exponential distribution with parameter $\nu(B)$.
(iii) $e(T_B)$ and T_B are independent, and for any Borel set A

$$\mathbb{P}(e(T_B)\in A)=\frac{\nu(A\cap B)}{\nu(B)}.$$

(iv) The process e' defined by $e'(t)=\xi$ if $e(t)\in B$ and $e'(t)=e(t)$ otherwise is a Poisson point process with characteristic measure $\nu\mathbf{1}_{B^c}$, and it is independent of $(T_B,e(T_B))$.

The process $(e(t),0\leq t\leq T_B)$ is called the process **stopped** at the first point in B; its law is characterized by Proposition 1.

If we define a deterministic function on $E\times[0,\infty)$ by $H_t(y)=\mathbf{1}_{B\times(t_1,t_2]}(y,t)$ it is clear that

$$\mathbb{E}\left(\sum_{0\leq t<\infty}H_t(e(t))\right)=(t_2-t_1)\nu(B);$$

this is the building block on which the following important result is based.

Proposition 2. ***(The compensation formula)*** *Let $H = (H_t, t \geq 0)$ be a predictable process taking values in the space of nonnegative measurable functions on $E_\cup\{\xi\}$ and having $H_t(\xi) \equiv 0$. Then*

$$\mathbb{E}\left(\sum_{0\leq t<\infty} H_t(e(t))\right) = \mathbb{E}\left(\int_0^\infty dt \int_E H_t(y)\nu(dy)\right).$$

A second important result is called **the exponential formula**;

Proposition 3. *Let f be a complex-valued Borel function on $E_\cup\{\xi\}$ with $f(\xi) = 0$ and*

$$\int_E |1 - e^{f(y)}|\nu(dy) < \infty.$$

Then for any $t \geq 0$

$$\mathbb{E}\left(\exp\left\{\sum_{0\leq s\leq t} f(e(s))\right\}\right) = \exp\left\{-t\int_E (1 - e^{f(y)})\nu(dy)\right\}.$$

1.3 The Lévy–Itô Decomposition

It is important to get a probabilistic interpretation of the Lévy–Khintchine formula, and this is what this decomposition does. Fundamentally, it describes the way that the measure Π determines the structure of the jumps in the process. Specifically it states that X can be written in the form

$$X_t = \gamma t + \sigma B_t + Y_t,$$

where B is a standard Brownian motion, and Y is a Lévy process which is independent of B, and is "determined by its jumps", in the following sense. Let $\Delta = \{\Delta_t, t \geq 0\}$ be a Poisson point process on $\mathbb{R} \times [0, \infty)$ with characteristic measure Π, and note that since $\Pi\{x : |x| \geq 1\} < \infty$, then $\sum_{s\leq t} 1_{\{|\Delta_s|\geq 1\}}|\Delta_s| < \infty$ a.s. Moreover if we define

$$Y_t^{(2)} = \sum_{s\leq t} 1_{\{|\Delta_s|\geq 1\}}\Delta_s,\ t \geq 0$$

then it is easy to see that, provided $c = \Pi\{x : |x| \geq 1\} > 0$, $(Y_t^{(2)}, t \geq 0)$ is a compound Poisson process with jump rate c, step distribution $F(dx) = c^{-1}\Pi(dx)\mathbf{1}_{\{|x|\geq 1\}}$ and, by the exponential formula, Lévy exponent

$$\Psi^{(2)}(\lambda) = \int_{|x|\geq 1} \{1 - e^{i\lambda x}\}\Pi(dx).$$

If

$$I = \int (1 \wedge |x|)\Pi(dx) < \infty, \tag{1.3.1}$$

then, by considering the limit of $\sum_{s\le t} 1_{\{\varepsilon<|\Delta_s|<1\}}\,|\Delta_s|$ as $\varepsilon\downarrow 0$, we see that

$$\sum_{s\le t} 1_{\{|\Delta_s|<1\}}\,|\Delta_s| < \infty \text{ a.s. for each } t<\infty,$$

and in this case we set $Y_t = Y_t^{(1)} + Y_t^{(2)}$, where

$$Y_t^{(1)} = \sum_{s\le t} \Delta_s 1_{\{|\Delta_s|<1\}},\ t\ge 0,$$

is independent of $Y^{(2)}$. Clearly, in this case Y has bounded variation (on each finite time interval), and it's exponent is

$$\Psi^{(1)}(\lambda) = \int_{|x|<1} \{1-e^{i\lambda x}\}\Pi(dx).$$

In this case we can rewrite the Lévy–Khintchine formula as

$$\Psi(\lambda) = -i\delta\lambda + \frac{\sigma^2}{2}\lambda^2 + \Psi^{(1)}(\lambda) + \Psi^{(2)}(\lambda),$$

where $\delta = \gamma - \int_{|x|<1} x\Pi(dx)$ is finite, and the Lévy–Itô decomposition takes the form

$$X_t = \delta t + \sigma B_t + Y_t^{(1)} + Y_t^{(2)},\ t\ge 0, \tag{1.3.2}$$

where the processes $B, Y^{(1)}$ and $Y^{(2)}$ are independent. The constant δ is called the **drift coefficient** of X.

However, if $I=\infty$ then a.s. $\sum_{s\le t}|\Delta_s| = \infty$ for each $t>0$, and in this case we need to define $Y^{(1)}$ differently: in fact as the a.s. limit as $\varepsilon\downarrow 0$ of the compensated partial sums,

$$Y_{\varepsilon,t}^{(1)} = \sum_{s\le t} 1_{\{\varepsilon<|\Delta_s|\le 1\}}\Delta_s - t\int_{\varepsilon<|x|\le 1} x\Pi(dx).$$

It is clear that $\{Y_{\varepsilon,t}^{(1)}, t\ge 0\}$ is a Lévy process, in fact a compensated compound Poisson process with exponent

$$\Psi_\varepsilon^{(1)}(\lambda) = \int_{-\infty}^{\infty} \{1-e^{i\lambda x}+i\lambda x\}\mathbf{1}_{(\varepsilon<|x|<1)}\Pi(dx),$$

and hence a martingale. The key point, (see e.g. [12] p14), is that the basic assumption that $\int(1\wedge x^2)\Pi(dx)<\infty$ allows us to use a version of Doob's maximal inequality for martingales to show that the limit as $\varepsilon\downarrow 0$ exists, has stationary and independent increments, and is a Lévy process with exponent

$$\Psi^{(1)}(\lambda) = \int_{-\infty}^{\infty} \{1-e^{i\lambda x}+i\lambda x\}\mathbf{1}_{(|x|<1)}\Pi(dx).$$

In this case the Lévy–Itô decomposition takes the form

$$X_t = \gamma t + \sigma B_t + Y_t^{(1)} + Y_t^{(2)},\ t \geq 0, \tag{1.3.3}$$

where again the processes $B, Y^{(1)}$ and $Y^{(2)}$ are independent.

Since $Y^{(2)}$ has unbounded variation we see that X has bounded variation $\iff \sigma = 0$ and $I < \infty$. All the examples we have discussed have bounded variation, except for Brownian motion and stable processes with index $\in (1,2)$.

To conclude this section, we record some information about the asymptotic behaviour of the Lévy exponent.

Proposition 4. *(i) In all cases we have*

$$\lim_{|\lambda|\to\infty} \frac{\Psi(\lambda)}{\lambda^2} = \frac{\sigma^2}{2}.$$

(ii) If X has bounded variation and drift coefficient δ,

$$\lim_{|\lambda|\to\infty} \frac{\Psi(\lambda)}{\lambda} = -i\delta.$$

(iii) X is a compound Poisson process if and only if Ψ is bounded.

(Note that we reserve the name compound Poisson process for a Lévy process with a finite Lévy measure, no Brownian component and drift coefficient zero.)

1.4 Lévy Processes as Markov Processes

It is clear that any Lévy process has the simple Markov property in the stronger, spatially homogeneous form that, given $X_t = x$, the process $\{X_{t+s}, s \geq 0\}$ is independent of $\{X_u, u < t\}$ and has the law of $\{x+X_s, s \geq 0\}$. In fact

- a similar form of the strong Markov property also holds. In particular this means that the above is valid if the fixed time t is replaced by a *first passage time*
$$T_B = \inf\{t \geq 0 : X_t \in B\}$$
whenever B is either open or closed.
- It is also the case that the semi-group of X has the Feller property and it turns out that the strong Feller property holds in the important special case that the law of X_t is absolutely continuous with respect to Lebesgue measure.
- In these, and some other circumstances, the resolvent kernel is absolutely continuous, i.e. there exists a non-negative measurable function $u^{(q)}$ such that

$$U^{(q)}f(x) := \int_0^\infty e^{-qt} P_t f(x) dt = \int_{-\infty}^\infty f(x+y) u^{(q)}(y) dy,$$

where

$$P_t f(x) = \mathbb{E}_x(f(X_t)).$$

- The associated potential theory requires no additional hypotheses; in particular if we write $X^* = -X$ for the dual of X we have the following duality relations. Let f and g be non-negative; then

$$\int_{\mathbb{R}} P_t f(x) g(x) dx = \int_{\mathbb{R}} f(x) P_t^* g(x) dx, \ t > 0,$$

and

$$\int_{\mathbb{R}} U^{(q)} f(x) g(x) dx = \int_{\mathbb{R}} f(x) U^{*(q)} g(x) dx, \ t > 0,$$

- The relation between X and X^* via time-reversal is also simple; *for each fixed* $t > 0$, the reversed process $\{X_{(t-s)-} - X_t, 0 \le s \le t\}$ and the dual process $\{X_s^*, 0 \le s \le t\}$ have the same law under $\mathbb{P}$.

In summary; X is a “nice” Markov process, and many of technical problems which appear in the general theory are simplified for Lévy processes.

2

Subordinators

2.1 Introduction

It is not difficult to see, by considering what happens near time 0, that a Lévy process which starts at 0 and only takes values in $[0, \infty)$ must have $\sigma = \Pi\{(-\infty, 0)\} = 0$, bounded variation and drift coefficient $\delta \geq 0$. Clearly such a process has monotone, non-decreasing paths. These processes, which are the continuous analogues of renewal processes, are called **subordinators**. (The name comes from the fact that whenever X is a Lévy process and T is an independent subordinator, the *subordinated* process defined by $Y_t = X_{T_t}$ is also a Lévy process.) Apart from the interest in subordinators as a sub-class of Lévy processes, we will see that they play a crucial rôle in fluctuation theory of general Lévy processes, just as renewal processes do in random-walk theory.

2.2 Basics

For subordinators it is possible, and convenient, to work with Laplace transforms rather than Fourier transforms. Since

$$\int_0^\infty (1 \wedge x)\Pi(dx) < \infty, \tag{2.2.1}$$

we can write the Lévy exponent in the form

$$\Psi(\lambda) = -i\delta\lambda + \int_0^\infty \{1 - e^{i\lambda x}\}\Pi(dx),$$

and it is clear from (2.2.1) that the integral converges on the upper half of the complex λ plane. So we can define the *Laplace exponent* by

$$\Phi(\lambda) = -\log \mathbb{E}\{e^{-\lambda X_1}\} = \Psi(i\lambda) = \delta\lambda + \int_0^\infty (1 - e^{-\lambda x})\Pi(dx), \tag{2.2.2}$$

and have

$$\mathbb{E}(e^{-\lambda X_t}) = \exp\{-t\Phi(\lambda)\}, \ \lambda \geq 0.$$

It is also useful to observe that, by integration by parts, we can rewrite (2.2.2) in terms of the Lévy tail, $\overline{\Pi}(x) = \Pi\{(x,\infty)\}$, as

$$\frac{\Phi(\lambda)}{\lambda} = \delta + \int_0^\infty \overline{\Pi}(x)e^{-\lambda x}dx. \tag{2.2.3}$$

A further integration by parts gives

$$\frac{\Phi(\lambda)}{\lambda^2} = \int_0^\infty e^{-\lambda x}\left\{\delta + I(x)\right\}dx, \tag{2.2.4}$$

where $I(x) = \int_0^x \overline{\Pi}(y)dy$ denotes the integrated tail of the Lévy measure.

One reason why subordinators are interesting is that they often turn up whilst studying other processes: for example, the first passage process in Brownian motion is a subordinator with $\delta = 0$ and $\Pi(dx) = cx^{-\frac{3}{2}}\mathbf{1}_{\{x>0\}}dx$, $\Phi(\lambda) = c'\lambda^{\frac{1}{2}}$. This is a stable subordinator of index 1/2. For $\alpha \in (0,1)$ **a stable subordinator of index** α has Laplace exponent

$$\Phi(\lambda) = c\lambda^\alpha = \frac{c\alpha}{\Gamma(1-\alpha)}\int_0^\infty (1-e^{-\lambda x})x^{-1-\alpha}dx.$$

The c here is just a scale factor, and the restriction on α comes from condition (2.2.1). Poisson processes are also subordinators, and the Gamma process we met earlier is a representative of the class of **Gamma subordinators**. These have

$$\Phi(\lambda) = a\log(1+b^{-1}\lambda) = \int_0^\infty (1-e^{-\lambda x})ax^{-1}e^{-bx}dx;$$

where $a, b > 0$ are parameters. (The second equality here is an example of the **Frullani integral**: see [20], Section 1.6.4.) This family is noteworthy because we also have an explicit expression for the distribution of X_t, viz

$$\mathbb{P}(X_t \in dx) = \frac{b^{at}}{\Gamma(at)}x^{at-1}e^{-bx}dx.$$

2.3 The Renewal Measure

Just as in the discrete case, an important object in the study of a subordinator is the associated renewal measure. Because X is transient, its potential measure

$$U(dx) = \mathbb{E}\left(\int_0^\infty \mathbf{1}_{\{X_t\in dx\}}dt\right) = \int_0^\infty \mathbb{P}(X_t \in dx)dt$$

is a Radon measure, and its distribution function, which we denote by $U(x)$, is called the renewal function of X. If $T_x = T_{(x,\infty)}$ we can also write

$$U(x) = U([0, x]) = \mathbb{E}T_x. \tag{2.3.1}$$

Let us first point out why the name is appropriate.

Lemma 1. *Let $Y = X_{\boldsymbol{e}}$, where $\boldsymbol{e}$ is an independent, Exp(1) random variable, and with $Y_1, Y_2 \cdots$ independent and identically distributed copies of Y, put $S_0 = 0$ and $S_n = \sum_1^n Y_j$ for $n \geq 1$. Write V for the renewal function of the renewal process S, viz $V(x) = \sum_0^\infty P(S_n \leq x)$. Then*

$$V(x) = 1 + U(x), \; x \geq 0.$$

Proof. Since

$$\begin{aligned} E(e^{-\lambda Y}) &= \int_0^\infty \int_0^\infty e^{-\lambda x} e^{-t} \mathbb{P}(X_t \in dx) dt \\ &= \int_0^\infty e^{-t} e^{-t\Phi(\lambda)} dt = \frac{1}{1 + \Phi(\lambda)} \end{aligned}$$

we see that

$$\int_0^\infty e^{-\lambda x} V(dx) = (1 - E(e^{-\lambda Y}))^{-1} = 1 + \frac{1}{\Phi(\lambda)}.$$

But

$$\begin{aligned} \int_0^\infty e^{-\lambda x} U(dx) &= \int_0^\infty e^{-\lambda x} \int_0^\infty \mathbb{P}(X_t \in dx) dt \\ &= \int_0^\infty e^{-t\phi(\lambda)} dt = \frac{1}{\Phi(\lambda)}. \end{aligned}$$

■

This tells us that asymptotic results such as the Renewal Theorem have analogues for subordinators: note in this context that Y has the same mean as X_1. Also, it is easy to see that, in essence, we don't need to worry about the difference between the lattice and non-lattice cases: the only time the support of U is contained in a lattice is when X is a compound Poisson process whose step distribution is supported by a lattice. If X is not compound Poisson, then the measure U is diffuse, and $U(x)$ is continuous; this is also true in the case of a compound Poisson process whose step distribution is diffuse, except that there is a Dirac mass at zero.

Another property which goes over to the continuous case is that of subadditivity, since the useful inequality

$$U(x + y) \leq U(x) + U(y), \; x, y \geq 0,$$

can be seen directly from (2.3.1). The behaviour of U for both large and small x is of interest, and in this the following lemma, which is slightly more general than we need, is useful.

Lemma 2. *Suppose that for* $\lambda > 0$

$$f(\lambda) = \lambda \int_0^\infty e^{-\lambda y} W(y)dy = \int_0^\infty e^{-y} W(y/\lambda)dy, \tag{2.3.2}$$

where W *is non-negative, non-decreasing, and such that there is a positive constant* c *with*

$$W(2x) \leq cW(x) \text{ for all } x > 0. \tag{2.3.3}$$

Then

$$W(x) \approx f(1/x), \tag{2.3.4}$$

where $\approx$ *means that the ratio of the two sides is bounded above and below by positive constants for all* $x > 0$.

Proof. It is immediate from (2.3.2) that for any $k > 0, \lambda > 0$,

$$W(k/\lambda) = e^k W(k/\lambda) \int_k^\infty e^{-y} dy \leq e^k \int_k^\infty e^{-y} W(y/\lambda)dy \leq e^k f(\lambda), \tag{2.3.5}$$

and with $k = 1$ this is one of the required bounds. Next, condition (2.3.3) gives

$$f(\lambda/2) = \int_0^\infty e^{-y} W(2y/\lambda)dy \leq c \int_0^\infty e^{-y} W(y/\lambda)dy = cf(\lambda).$$

Using this and rewriting (2.3.5) as

$$W(y/\lambda) = W((y/2)/(\lambda/2)) \leq e^{y/2} f(\lambda/2)$$

gives, for any $x > 0$,

$$\begin{aligned} f(\lambda) &\leq W(x/\lambda) \int_0^x e^{-y} dy + f(\lambda/2) \int_x^\infty e^{y/2} e^{-y} dy \\ &= (1 - e^{-x}) W(x/\lambda) + 2f(\lambda/2) e^{-x/2} \\ &\leq (1 - e^{-x}) W(x/\lambda) + 2cf(\lambda) e^{-x/2}. \end{aligned}$$

Assuming, with no loss of generality, that $c > 1/4$, and choosing $x = x_0 := 2 \log 4c$ and an integer n_0 with $2^{n_0} \geq x_0$ we deduce, using (2.3.3) again, that

$$f(\lambda) \leq 2 \left(1 - \frac{1}{16c^2}\right) W(x_0/\lambda) \leq 2c^{n_0} \left(1 - \frac{1}{16c^2}\right) W(1/\lambda),$$

and this is the other bound. ■

For some applications, it is important that the constants in the upper and lower bounds only depend on W through the constant c in (2.3.3). For example, when $c = 2$, as it does in the special case that W is subadditive, we can take them to be $8/63$ and e.

Corollary 1. *Let X be any subordinator, and write $I(x) = \int_0^x \overline{\Pi}(y)dy$. Then*

$$U(x) \approx \frac{1}{\Phi(1/x)} \text{ and } \frac{\Phi(x)}{x} \approx I(1/x) + \delta.$$

Proof. Recall (2.2.4) and the fact that $\int_0^\infty e^{-\lambda x} U(x)dx = \lambda/\phi(\lambda)$ and check that the conditions of the previous lemma are satisfied. ∎

These estimates can of course be refined if we assume more. If either of U or Φ is in $RV(\alpha)$ (i.e. is regularly varying with index α; see [20] for details) with $\alpha \in [0,1]$ at $0+$ or ∞, then the other is in $RV(\alpha)$ at ∞, respectively $0+$; in fact

$$\Gamma(1+\alpha)U(x) \sim \frac{1}{\Phi(1/x)}.$$

Similarly we have

$$\Gamma(2-\alpha)\{I(x)+\delta\} \sim x\Phi(1/x),$$

and moreover when this happens with $\alpha < 1$, the monotone density theorem applies and

$$\Gamma(1-\alpha)\overline{\Pi}(x) \sim \frac{1}{\Phi(1/x)}.$$

2.4 Passage Across a Level

We will be interested in the undershoot and overshoot when the subordinator crosses a positive level x, but in continuous time we have to consider the possibility of continuous passage, i.e. that T_x is not a time at which X jumps. We start with our first example of the use of the compensation formula.

Theorem 2. *If X is a subordinator we have*

(i) for $0 \le y \le x$ and $z > x$

$$\mathbb{P}(X_{T_x -} \in dy, X_{T_x} \in dz) = U(dy)\Pi(dz-y):$$

(ii) for every $x > 0$,

$$\mathbb{P}(X_{T_x -} < x = X_{T_x}) = 0.$$

Proof. (i) Recall that the process of jumps Δ is a Poisson point process on $\mathbb{R}\times[0,\infty)$ with characteristic measure Π, so

$$\begin{aligned}\mathbb{P}(X_{T_x -} \in dy, X_{T_x} \in dz) &= \mathbb{E}\left(\sum_{t\ge 0} \mathbf{1}_{(X_{t-}\in dy, X_t \in dz)}\right)\\ &= \mathbb{E}\left(\sum_{t\ge 0} \mathbf{1}_{(X_{t-}\in dy, \Delta_t \in dz-y)}\right)\end{aligned}$$

$$= \int_0^\infty dt \mathbb{E}\left(\mathbf{1}_{(X_{t-}\in dy)} \int_{-\infty}^{\infty} \Pi(ds)\mathbf{1}_{(s\in dz-y)}\right)$$
$$= \int_0^\infty dt \mathbb{P}(X_t \in dy)\Pi(dz-y) = U(dy)\Pi(dz-y).$$

(ii) The statement is clearly true if X is a compound Poisson process, since then the values of X form a discrete set, and otherwise we know that U is diffuse. In this case the above argument gives

$$\mathbb{P}(X_{T_x-} < x = X_{T_x}) = \int_{[0,x)} U(dy)\Pi(\{x-y\}) = 0,$$

since $\Pi(\{z\}) = 0$ off a countable set. ■

Observe that a similar argument gives the following extension of (i):

$$\mathbb{P}(X_{T_x-} \in dy, X_{T_x} \in dz, T_x \le t) = \int_0^t \mathbb{P}(X_s \in dy) ds \Pi(dz-y).$$

From this we deduce the following equality of measures:

$$\mathbb{P}(X_{T_x-} \in dy, X_{T_x} \in dz, T_x \in dt) = \mathbb{P}(X_t \in dy)\Pi(dz-y)dt$$
$$\text{for } 0 \le y \le x, z > x \text{ and } t > 0.$$

Part (ii) says that if a subordinator crosses a level by a jump, then a.s. that jump takes it over the level.

It turns out that the question of continuous passage (or "creeping") of a subordinator is quite subtle, and was only resolved in [58], and we refer to that paper, [22] or [12], Section III.2 for a proof of the following.

Theorem 3. *If X is a subordinator with drift δ,*

(i) if $\delta = 0$ then $\mathbb{P}(X_{T_x} = x) = 0$ for **all** $x > 0$,
(ii) if $\delta > 0$ then U has a strictly positive and continuous density u on $(0, \infty)$,

$$\mathbb{P}(X_{T_x} = x) = \delta u(x) \text{ for } \textbf{all } x > 0, \tag{2.4.1}$$

and $\lim_{x\downarrow 0} u(x) = 1/\delta$.

Parts of this are easy; for example, by applying the strong Markov property at time T_x we get

$$U(dw) = \int_{[x,w]} U(dw-z)\mathbb{P}(X_{T_x} \in dz), \ w \ge x,$$

and taking Laplace transforms gives

$$\int_{[x,\infty)} e^{-\lambda w} U(dw) = \int_{[0,\infty)} e^{-\lambda w} U(dw) \int_{[x,\infty)} e^{-\lambda z} \mathbb{P}(X_{T_x} \in dz)$$
$$= \frac{\mathbb{E}(e^{-\lambda X_{T_x}})}{\Phi(\lambda)}.$$

This leads quickly to

$$\int_0^\infty e^{-qx}\mathbb{E}\left(e^{-\lambda(X_{T_x}-x)}\right)dx = \frac{\Phi(\lambda)-\Phi(q)}{(\lambda-q)\Phi(q)}, \tag{2.4.2}$$

and since, by Proposition 4, Chapter 1, $\lambda^{-1}\Phi(\lambda)\to\delta$ as $\lambda\to\infty$, we arrive at the conclusion that

$$\int_0^\infty e^{-qx}\mathbb{P}(X_{T_x}=x)dx = \frac{\delta}{\Phi(q)} = \delta\int_0^\infty e^{-qx}U(dx).$$

If $\delta=0$ this tells us that $\mathbb{P}(X_{T_x}=x)=0$ for a.e. Lebesgue x. Also, if $\delta>0$, then a simple Fourier-analytic estimate shows that U is absolutely continuous, and hence statement (2.4.1) holds a.e. The proof of the remaining statements in [12], Section III.2 is based on clever use of the inequalities:

$$\mathbb{P}(X_{T_{x+y}}=x+y)\geq\mathbb{P}(X_{T_x}=x)\mathbb{P}(X_{T_y}=y)$$
$$\mathbb{P}(X_{T_{x+y}}=x+y)\leq\mathbb{P}(X_{T_x}=x)\mathbb{P}(X_{T_y}=y)+1-\mathbb{P}(X_{T_x}=x).$$

Further results involving creeping of a general Lévy process will be discussed in Chapter 6.

2.5 Arc-Sine Laws for Subordinators

Our interest here is in the analogue of the "arc-sine theorem for renewal processes", see e.g. [20], Section 8.6. Apart from the interest in the results for subordinators per se, we will see that, just as in the case of random walks, it enables us to derive arc-sine theorems for general Lévy processes.

Note that the the random variable $x-X_{T_x-}$, which we have referred to as the undershoot, is the analogue of the quantity referred to in Renewal theory as, "unexpired lifetime" or "backward recurrence time", but we will phrase our results in terms of X_{T_x-}. First we use an argument similar to that leading to (2.4.2) to see that

$$\int_0^\infty e^{-qx}\mathbb{E}\left(e^{-\lambda X_{T_x-}}\right)dx = \frac{\Phi(q)}{q\Phi(q+\lambda)},$$

and hence, writing $A_t(x)=x^{-1}X(T_{tx}-)$

$$\int_0^\infty e^{-qt}\mathbb{E}\left(e^{-\lambda A_t(x)}\right)dt = \frac{\Phi(q/x)}{q\Phi((q+\lambda)/x)}.$$

Now if X is a stable subordinator with index $0<\alpha<1$, we see that the right-hand side does not depend on x, and equals $q^{\alpha-1}(q+\lambda)^{-\alpha}$. By checking that

$$\int_0^\infty e^{-qt}\int_0^t e^{-\lambda s}\frac{s^{\alpha-1}(t-s)^{-\alpha}}{\Gamma(\alpha)\Gamma(1-\alpha)}ds = q^{\alpha-1}(q+\lambda)^{-\alpha}$$

we see that for each $t, x > 0$, $A_t(x) \stackrel{D}{=} A_t(1) \stackrel{D}{=} A_1(1)$, and this last has the generalised arc-sine law with parameter α. As a general subordinator X is in the domain of attraction of a standard stable subordinator of index α (i.e. $\exists$ a norming function $b(t)$ such that the process $\{X_{ts}/b(t), s \geq 0\}$ converges weakly to it), as $t \to \infty$ or $t \to 0+$, if and only if its exponent $\Phi \in RV(\alpha)$ (at 0 or ∞, respectively), the following should not be a surprise. For a proof we again refer to [12], Section III.3.

Theorem 4. *The following statements are equivalent.*

(i) The random variables $x^{-1}X(T_x-)$ converge in distribution as $x \to \infty$ (respectively as $x \to 0+$).
(ii) $\lim x^{-1}\mathbb{E}(X(T_x-)) = \alpha \in [0,1]$ as $x \to \infty$ (respectively as $x \to 0+$).
(iii) The Laplace exponent $\Phi \in RV(\alpha)$ (at 0 or ∞, respectively) with $\alpha \in [0,1]$.

When this happens the limit distribution is the arc-sine law with parameter α if $0 < \alpha < 1$, and is degenerate at 0 or 1 if $\alpha = 0$ or 1.

2.6 Rates of Growth

The following fundamental result shows that strong laws of large numbers hold, both at infinity and zero.

Proposition 5. *For any subordinator X*

$$\lim_{t\to\infty} \frac{X_t}{t} \stackrel{a.s.}{=} \mathbb{E}X_1 = \delta + \int_0^\infty \overline{\Pi}(x)dx \leq \infty, \qquad \lim_{t\to 0+} \frac{X_t}{t} \stackrel{a.s.}{=} \delta \geq 0.$$

Proof. The first result follows easily by random-walk approximation, and the second follows because we know from $\lim_{t\to 0+} t\Phi(\lambda/t) = \delta\lambda$ that we have convergence in distribution, and ([12], Section III.4) we can also show that $(t^{-1}X_t, t > 0)$ is a reversed martingale. ∎

There are many results known about rates of growth of subordinators, both for large and small times. Just to give you an indication of their scope I will quote a couple of results from [12], Section III.4.

Theorem 5. *Assume that $\delta = 0$ and $h : [0,\infty) \to [0,\infty)$ is a non-decreasing function such that $t^{-1}h(t)$ is also non-decreasing. Then*

$$\limsup_{t\to 0+} \frac{X_t}{h(t)} = \infty \ a.s.$$

if and only if

$$\int_0^1 \overline{\Pi}(h(x))dx < \infty,$$

and if these fail,

$$\lim_{t \to 0+} \frac{X_t}{h(t)} = 0 \ a.s.$$

Notice that in the situation of this result, the lim sup has to be either 0 or ∞; this contrasts with the behaviour of the lim inf, as we see from the following.

Theorem 6. *Suppose that $\Phi \in RV(\alpha)$ at ∞, and Φ has inverse ϕ. Define*

$$f(t) = \frac{\log|\log t|}{\phi(t^{-1}\log|\log t|)}, \ 0 < t < 1/e.$$

Then

$$\liminf \frac{X_t}{f(t)} = \alpha(1-\alpha)^{(1-\alpha)/\alpha} \ a.s. \ .$$

There are exactly analogous statements for large t.

2.7 Killed Subordinators

It is important, particularly in connection with the ladder processes, to treat subordinators with a possibly finite lifetime. In order for the Markov property to hold, the lifetime has to be exponentially distributed, say with parameter k. It is also easy to see that if $\tilde{X}$ is such a subordinator, then it can be considered as a subordinator X with infinite lifetime killed at an independent exponential time, and that the corresponding exponents are related by

$$\tilde{\Phi}(\lambda) = k + \Phi(\lambda), \ \lambda \geq 0.$$

So the **characteristics** of a (possibly killed) subordinator are its Lévy measure Π, its drift coefficient δ, and its killing rate $k \geq 0$.

3

Local Times and Excursions

3.1 Introduction

A key idea in the study of Lévy processes is that of "excursions away from the maximum", which we can also describe as excursions away from zero of the reflected process

$$R = S - X, \text{ where } S_t = \sup\{0 \vee X_s; 0 \le s \le t\}.$$

Now it can be shown that R is a strong Markov process, (see [12], p. 156), so the natural way to study its zero set is through a local time. So here we briefly review these concepts for a general Markov process M. It is easy to think of examples where such a process, starting from 0,

(i) does not return to 0 at arbitrarily small times;
(ii) remains at 0 for a positive time; or
(iii) leaves 0 instantaneously but returns to 0 at arbitrarily small times.

We have to treat these three cases separately, but the third case is the most interesting one.

3.2 Local Time of a Markov Process

Let $(\Omega', \mathcal{G}, \mathbf{P})$ be a probability space satisfying the usual conditions and $M = (M_t, t \ge 0)$ a process taking values in $\mathbb{R}$ with cadlag paths such that $\mathbf{P}(M_0 = 0) = 1$. Suppose further there is a family $(\mathbf{P}_x, x \in \mathbb{R})$ of probability measures which correspond to the law of M starting from x, for which the following version of the strong Markov property holds:

For every stopping time $T < \infty$, under the conditional law $\mathbf{P}(\cdot | M_T = x)$, the shifted process $(M_{T+t}, t \ge 0)$ is independent of $\mathcal{G}_T$ and has the law $\mathbf{P}_x$.

This entails the Blumenthal zero–one law, so the σ-field $\mathcal{G}_0$ is trivial, and we can formalise the trichotomy referred to above as follows. We know that $r_T := \inf\{t > T : M_t = 0\}$ is also a stopping time when T is, in particular the first return time r_0 is a $\mathcal{G}_0$-measurable stopping time, so $\mathbf{P}(r_0 = 0)$ is 1 or 0. We say that 0 is **regular** or **irregular** (for 0) according as it is 1 or 0. In the regular case we introduce the first exit time $s_1 = \inf\{t \geq 0 : M_t \neq 0\}$, which is also a $\mathcal{G}_0$-measurable stopping time, and we say that 0 is a **holding point** if $\mathbf{P}(s_1 = 0) = 0$, and an **instantaneous point** if $\mathbf{P}(s_1 = 0) = 1$.

3.3 The Regular, Instantaneous Case

There are several different approaches to the construction of local time; here I outline the direct approach based on approximations involving the numbers of excursion intervals of certain types given in [12], Section IV.2.

The zero set of M, $\mathcal{Z} = \{t : M_t = 0\}$ and its closure $\overline{\mathcal{Z}}$ play central rôles. $\mathcal{Z}$ is an example of a regenerative set; informally this means that if we take a typical point of $\mathcal{Z}$ as a new origin the part of it to the right has the same probabilistic structure as $\mathcal{Z}$, and is independent of the part to the left.

An open interval (g, d) with $M_t \neq 0$ for all $g < t < d$, $g \in \overline{\mathcal{Z}}$ and $d \in \overline{\mathcal{Z}}_{\cup}\{\infty\}$ is called an **excursion interval**; these intervals are also those that arise in the canonical decomposition of the open set $[0, \infty) - \overline{\mathcal{Z}}$.

Let $l_n(a), g_n(a)$ and $d_n(a)$ denote the length, left-hand end-point and right-hand endpoint of the nth excursion interval whose length exceeds a, and introduce a non-increasing and right-continuous function $\overline{\mu}$ to describe the distribution of lengths of excursions by

$$\overline{\mu}(a) = \begin{cases} 1/\mathbf{P}(l_1(a) > c) & \text{if } a \leq c, \\ \mathbf{P}(l_1(c) > a) & \text{if } a > c. \end{cases}$$

Here c has been chosen so that $\mathbf{P}(l_1(a) > c) > 0$ for all $a \leq c$, which is always possible. Let

$$N_a(t) = \sup\{n : g_n(a) < t\},$$

which is the number of excursions with length exceeding a which start before t. Then the main result is

Theorem 7. *The following statements hold a.s.*

(i) *For all $t \geq 0$, $N_a(t)/\overline{\mu}(a)$ converges as $a \to 0+$; denote its limit by $L(t)$.*
(ii) *The mapping $t \to L(t)$ is increasing and continuous.*
(iii) *The support of the Stieltjes measure dL is $\overline{\mathcal{Z}}$.*
Also
(iv) *L is adapted to the filtration $\mathcal{G}$.*
(v) *For every a.s. finite stopping time T with $M_T = 0$ a.s., the shifted process $\{(M_{T+t}, L(T+t) - L(T)), t \geq 0\}$ is independent of $\mathcal{G}_T$ and has the same law as (M, L) under $\mathbf{P}$.*

(vi) *If L' is any other continuous increasing process such that the support of the Stieltjes measure dL' is contained in $\overline{\mathcal{Z}}$, and which has properties (iv) and (v), then for some constant $k \geq 0$ we have $L' \equiv kL$.*

The proof actually works by looking at the convergence of the ratio $N_a(d_1(u))/\overline{\mu}(a)$, and a byproduct of the proof is that

$$L(d_1(u)) \text{ is } \mathrm{Exp}(\overline{\mu}(u))\text{distributed and independent of } l_1(u). \tag{3.3.1}$$

If the set $\overline{\mathcal{Z}}_\cap[0,t]$ has positive Lebesgue measure, then the Lebesgue measure of this set would satisfy conditions (iv) and (v) of Theorem 7, and this is consistent with:

Corollary 2. *There exists a constant $\delta \geq 0$ such that, a.s.*

$$\int_0^t \mathbf{1}_{\{M_s=0\}} ds = \int_0^t \mathbf{1}_{\{s \in \overline{\mathcal{Z}}\}} ds = \delta L(t) \text{ for all } t \geq 0. \tag{3.3.2}$$

Next we see the relevance of subordinators in this setting. We study L via its right continuous inverse

$$L^{-1}(t) = \inf\{s \geq 0 : L(s) > t\};$$

note that

$$L^{-1}(t-) := \lim_{s \uparrow t} L^{-1}(s) = \inf\{s \geq 0 : L(s) \geq t\}.$$

It can be shown that these are both stopping times, that the process L^{-1} is adapted to the filtration $\{\mathcal{G}_{L^{-1}(t)}; t \geq 0\}$, and that

$$L^{-1}(L(t)) = \inf\{s > t : M_s = 0\},$$
$$L^{-1}(L(t)-) = \sup\{s < t : M_s = 0\}$$

coincide with the left and right-hand end-points of the excursion interval containing t.

Since we have constructed the process by approximation, and in the discrete case the analogue of L is the process which counts the number of returns to 0 by time t, the inverse of which is a renewal process, the following result is very natural.

Theorem 8. *The inverse local time process $L^{-1} = (L^{-1}(t), t \geq 0)$ is a (possibly killed) subordinator with Lévy measure μ, drift coefficient δ, and killing rate $\overline{\mu}(\infty)$. Its exponent is given by*

$$\Phi(\lambda) = \overline{\mu}(\infty) + \lambda \left(\delta + \int_0^\infty e^{-\lambda x} \overline{\mu}(x) dx \right),$$

where $\overline{\mu}(x) = \mu\{(x, \infty)\}$.

The main steps in the proof of this when $\overline{\mu}(\infty) = 0$ are

- The shifted process $\widetilde{M} = \{M_{L^{-1}(t)+s}, s \geq 0\}$ has local time given by $\widetilde{L}(s) = L(L^{-1}(t) + s\} - t$, and hence

$$\widetilde{L}^{-1}(s) = L^{-1}(t+s) - L^{-1}(t).$$

  This implies that L^{-1} is a subordinator.
- We can identify the Lévy measure of this subordinator with μ by using (3.3.1).
- The jumps in L^{-1} correspond to the lengths of the excursion intervals, so $L^{-1}(t)$ is the sum of the lengths of the excursions completed by local time t plus the time spent at 0, so by (3.3.2),

$$\begin{aligned} L^{-1}(t) &= \int_0^{L^{-1}(t)} \mathbf{1}_{\{s \in \overline{\mathcal{Z}}\}} ds + \sum_{s \leq t} \Delta L^{-1}(s) \\ &= \delta L(L^{-1}(t)) + \sum_{s \leq t} \Delta L^{-1}(s) \\ &= \delta t + \sum_{s \leq t} \Delta L^{-1}(s). \end{aligned}$$

  This identifies the drift as δ.

It is not difficult to see that the case of a killed subordinator, when $\overline{\mu}(\infty) > 0$, corresponds exactly to the case that 0 is transient, so there exists an excursion of infinite length, and the case $\overline{\mu}(\infty) = 0$ corresponds exactly to the case that 0 is recurrent.

Finally it should be remarked that subordinators, inverse local times for Markov processes, and regenerative sets are inextricably connected; for example every subordinator is the inverse local time for some Markov process.

3.4 The Excursion Process

How can we describe the excursions away from zero of M, that is the pieces of path of the form $\{M_{g+t}, 0 \leq t < d - g\}$? These take values in excursion space $\mathcal{E} = \cup_{a>0} \mathcal{E}^{(a)}$, where

$$\mathcal{E}^{(a)} = \{\omega \in \Omega : \zeta > a \text{ and } \omega(t) \neq 0 \text{ for all } 0 < t < \zeta\},$$

and ζ is the lifetime of an excursion, which corresponds to $d-g$. The excursions whose lifetimes exceed $a > 0$ clearly form an independent and identically distributed sequence, and we can define a σ-finite measure on $\mathcal{E}$ by putting

$$n(\cdot | \zeta > a) = \mathbf{P}\{(M_{g_1(a)+t}, 0 \leq t < l_1(a)) \in \cdot).$$

One can check that $n(\zeta > a) = \overline{\mu}(a)$, so for general Λ

$$n(\Lambda) = \lim_{a \downarrow 0} \overline{\mu}(a) n(\Lambda | \zeta > a).$$

We can see that under n, conditionally on $\{\omega(a) = x, a < \zeta\}$, the shifted process $\{\omega(a+t), 0 \leq t < \zeta - a\}$ is independent of $\{\omega(t), 0 \leq t < a\}$, and is distributed as $\{M_t, 0 \leq t < r_0\}$ under $\mathbf{P}_x$. In particular the **excursion measure** n has the simple Markov property.

Now we introduce the **excursion process** $e = (e(t), t \geq 0)$, where we put $l(t) = L^{-1}(t) - L^{-1}(t-)$ and

$$e(t) = \begin{cases} \{M_{L^{-1}(t-)+s}, 0 \leq s < l(t)\} & \text{if } l(t) > 0, \\ \xi & \text{if } l(t) = 0, \end{cases}$$

and ξ is an additional isolated point. The following result is essentially due to Itô [56].

Theorem 9. *(i) If 0 is recurrent, then e is a Poisson point process with characteristic measure n.*

(ii) If 0 is transient, then $\{e(t), 0 \leq t \leq L(\infty)\}$ is a Poisson point process with characteristic measure n, stopped at the first point in $\mathcal{E}^{(\infty)}$, the set of excursions of infinite length.

This allows us to use the techniques of Poisson point processes to carry out explicit calculations; in particular we can rewrite the compensation formula as follows. For every left-hand end-point $g < \infty$ of an excursion interval, denote by $\varepsilon_g = \{M_{g+t}, 0 \leq t < d - g\}$ the excursion starting at time g. Consider a measurable function $F : \mathbb{R}_+ \times \Omega' \times \mathcal{E} \to [0, \infty)$ which is such that for every $\varepsilon \in \mathcal{E}$, the process $t \to F_t(\varepsilon) = F(t, \omega', \varepsilon)$ is left- continuous and adapted. Then

$$\sum_g F_g(\varepsilon_g) = \sum_t F_{L^{-1}(t)}(e_t) \mathbf{1}_{\{t \leq L(\infty)\}}$$

and we deduce that

$$\mathbf{E}\left(\sum_g F_g(\varepsilon_g)\right) = \mathbf{E}\left(\int_0^\infty dL(s) \left[\int_{\mathcal{E}} F_s(\varepsilon) n(d\varepsilon)\right]\right). \tag{3.4.1}$$

For some examples of applying this result, see [12], p. 120.

3.5 The Case of Holding and Irregular Points

In the case of 0 being a holding point, things are much simpler, as there is a sequence of exit/entrance times, $0 = r_0 < s_1 < r_1 \cdots$, where $r_n = \inf\{t > s_n : M_t = 0\}$, $s_n = \inf\{t > r_n : M_t \neq 0\}$. We have $M_{s_1} \neq 0$ a.s., and s_1 has an exponential distribution and is independent of the first excursion

$\{M_{s_1+t}, 0 \leq t < r_1 - s_1\}$. On the event $\{r_1 < \infty\}$ we have $M_{r_1} = 0$ a.s., and we can repeat the argument to see that the zero set can be expressed as

$$\mathcal{Z} = [r_0, s_1)\cup[r_1, s_2)\cup \cdots .$$

In this case we can take $L(t)$ to be proportional to the occupation process,

$$L(t) = \delta \int_0^t \mathbf{1}_{\{M_s=0\}} ds,$$

where $\delta > 0$, and n to be a finite measure proportional to the law of the first excursion of M, viz $\{M_{s_1+t}, 0 \leq t < r_1 - s_1\}$. Then again L^{-1} is a subordinator (possibly killed) with drift coefficient δ, and the excursion process is a Poisson point process with finite characteristic measure n.

In the case that 0 is irregular it is clear the successive return times to 0 form a sequence $r = \{r_n, n \geq 0\}$ which is in fact an increasing random walk, i.e. a renewal process. Again the process of excursions is an independent and identically distributed sequence, and again we can take n to be a finite measure proportional to the law of the first excursion of M. The natural definition of L is as the process that counts the number of returns to 0, and then its inverse would be r, which is a discrete time process. The solution to this problem is to transform r by an independent Poisson process of unit rate, which leads to the definition of L by

$$L(t) = \sum_0^{n(t)} \boldsymbol{e}_j, \text{ where } n(t) = \max(n : r_n \leq t),$$

and $\boldsymbol{e}_1, \boldsymbol{e}_2, \cdots$ are independent unit rate Exponential random variables, independent of M. Of course L is only right-continuous, and we have to augment the filtration $\mathcal{G}$ to make L adapted, but with this definition L^{-1} is again a subordinator and the excursion process is again a Poisson point process.

4

Ladder Processes and the Wiener–Hopf Factorisation

4.1 Introduction

It was shown by Spitzer, Baxter, Feller and others that the ladder processes are absolutely central to the study of fluctuation theory in discrete time, and we will see that the same is true in continuous time. However a first difficulty in setting up the corresponding theory is that the times at which a Lévy process X attains a new maximum do not, typically, form a discrete set. This means that a basic technique in random-walk theory which consists of splitting a path at the first time it takes a positive value, is not applicable. Also Feller showed that Wiener–Hopf results for random walks are fundamentally combinatorial results about the paths, and it doesn't seem possible to apply such methods to paths of Lévy processes.

In the early days the only way round these difficulties was to use very analytic methods and/or random-walk approximation. But now, as far as possible, we prefer to use sample-path arguments, excursion theory and local time techniques: but it is impossible to avoid analytical methods altogether.

We will start with a short review of Wiener–Hopf factorisation for random walks: more details can be found in Chapter XII of [47], or Section 8.9 of [20].

Most of the material in the rest of this Chapter is in Chapter VI of [12].

4.2 The Random Walk Case

Let $Y_1, Y_2, \cdots$ be independent and identically distributed real-valued random variables with distribution F. The process $S = (S_n, n \geq 0)$ where $S_0 \equiv 0$ and $S_n = \sum_1^n Y_r$ for $n \geq 1$ is called a random walk with step distribution F. In the special case that $F((-\infty, 0)) = 0$, S is called a renewal process. For convenience, we will assume that F has no atoms, so that $P\{S_n = x$ for some $n \geq 1\} = 0$ for all x : this means we don't have to distinguish between strong and weak ladder variables in the following.

Define $T^{\pm} = (T_n^{\pm}, n \geq 0)$ and $H^{\pm} = (H_n^{\pm}, n \geq 0)$, where $H_n^{\pm} = |S_{T_n^{\pm}}|$ and

$$T_0^{\pm} \equiv 0,\ T_{n+1}^{\pm} = \min(r : \pm S_r > H_n^{\pm}),\ n \geq 0.$$

Each of these processes are renewal processes: the *increasing and decreasing ladder time and ladder height processes.*

The connection between the distributions of these processes and F is given analytically by the following identity, which is due to Baxter. It is the discrete version of Fristedt's formula: see Theorem 10 in the next section.

$$1 - E(r^{T_1^+} e^{itH_1^+}) = \exp - \sum_1^{\infty} \frac{r^n}{n} E(e^{itS_n} : S_n > 0). \tag{4.2.1}$$

From this, and the analogous result for the decreasing ladder variables, the discrete version of the Wiener–Hopf factorisation follows:

$$1 - rE(e^{itY_1}) = \left[1 - rE(r^{T_1^+} e^{itH_1^+})\right] \left[1 - rE(r^{T_1^-} e^{-itH_1^-})\right]. \tag{4.2.2}$$

These results have several immediate corollaries, some of which we list below.

- The Wiener–Hopf factorisation of the characteristic function is got by putting $r = 1$ in (4.2.2):

$$1 - E(e^{itY_1}) = \left[1 - E(e^{itH_1^+})\right] \left[1 - E(e^{-itH_1^-})\right]. \tag{4.2.3}$$

- Spitzer's formula

$$1 - E(r^{T_1^+}) = \exp - \sum_1^{\infty} \frac{r^n}{n} P(S_n > 0)$$

is the special case $t = 0$ of (4.2.1).
- $S_n \overset{a.s.}{\to} -\infty \iff P(T_1^+ = \infty) > 0 \iff \sum_1^{\infty} \frac{1}{n} P(S_n > 0) < \infty$, $S_n \overset{a.s.}{\to} \infty \iff P(T_1^- = \infty) > 0 \iff \sum_1^{\infty} \frac{1}{n} P(S_n < 0) < \infty$;
S oscillates$\iff$ both T_1^+ and T_1^- are proper$\iff$ both series diverge.
- In the case of oscillation, $ET_1^+ = T_1^- = \infty$; if $S_n \overset{a.s.}{\to} \infty$ then $ET_1^+ < \infty$, and $EH_1^+ = ET_1^+ EY_1$ if $E|Y_1| < \infty$.

Remark 1. *A simple proof of these results can be based on Feller's lemma, which is a purely combinatorial result. It says that if $(0, s_1, s_2, \cdots s_n)$ is a deterministic path based on steps $y_r = s_r - s_{r-1}$, $r = 1, 2, \cdots n$, then provided $s_n > 0$, in the set of n paths we get by cycically permuting the $y's$, there is always at least one in which n is an increasing ladder time; moreover if there are k such paths, then in each of them there are exactly k increasing ladder times. From this we can deduce the identity*

$$\frac{P(S_n \in dx)}{n} = \sum_1^{\infty} \frac{1}{k} P(T_k^+ = n, H_k^+ \in dx),\ n \geq 1, x > 0, \tag{4.2.4}$$

and this is fully equivalent to (4.2.1). (See Proposition 8 in Chapter 5 for the Lévy process version of (4.2.4).)

Finally, in this setting time reversal gives the following useful result, which is often referred to as the duality lemma: let $U^{\pm}(dx) = \sum_1^\infty P(H_k^{\pm} \in dx)$ be the renewal measures of $H^{\pm}$, then

$$U^+(dx) = \sum_1^\infty P(S_k \in dx, T_1^- > k). \tag{4.2.5}$$

An immediate consequence of this is the relation

$$P(H_1^+ \in dx) = \int_{-\infty}^0 F(y + dx) U^-(dy); \tag{4.2.6}$$

the Lévy process version of this has only been established recently: see Theorem 16 in Chapter 5.

4.3 The Reflected and Ladder Processes

The crucial idea is to think of the set of "increasing ladder times" of X as the zero set of the reflected process

$$R = S - X, \text{ where } S_t = \sup\{0 \vee X_s; 0 \le s \le t\}.$$

We have already mentioned that R is a strong Markov process, and that the natural way to study its zero set is through a local time. So, whenever 0 is regular for R, (i.e. X almost surely has a new maximum before time ε, for any $\varepsilon > 0$) we write $L = \{L_t, t \ge 0\}$, for a Markov local time for R at 0, $\tau = L^{-1}$ for the corresponding inverse local time, and $H = X(\tau) = S(\tau)$. Then τ and H are both subordinators, and we call them the (upwards) ladder time and ladder height processes of X. In fact the pair (τ, H) is a bivariate subordinator, as is (τ^*, H^*), the downwards ladder process, which we get by replacing X by $X^* = -X$ in the above. (We are using subordinator here in the extended sense; clearly if 0 is transient for R then τ and H are killed subordinators.) So the law of the ladder processes is characterized by

$$\mathbb{E}\left(e^{-(\alpha\tau_t + \beta H_t)}\right) = e^{-t\kappa(\alpha,\beta)}, \ \alpha, \beta \ge 0,$$

where, by an obvious extension of the real-valued case, κ has the form

$$\kappa(\alpha, \beta) = k + \eta\alpha + \delta\beta + \int_0^\infty \int_0^\infty \left\{1 - e^{-(\alpha x_1 + \beta x_2)}\right\} \mu(dx_1 dx_2)$$

with $k, \eta, \delta \ge 0$ and

$$\int_0^\infty \int_0^\infty (x_1 \wedge 1)(x_2 \wedge 1) \mu(dx_1 dx_2) < \infty.$$

One of our aims is to get more information about this Laplace exponent.

The connection between the distribution of the ladder processes and that of X can be formulated in various ways. All of these relate the distribution of a real-valued process to that of two processes taking non-negative values, and so can be thought of as versions of the Wiener–Hopf factorisation for X. The first of these is due to Pecherskii and Rogozin [79], who derived it by random-walk approximation. Let $G_t = \sup\{s \le t : S_s = X_s\}$. Then the identity is

$$\frac{q}{q + \lambda + \Psi(\theta)} = \Psi_q(\theta, \lambda)\Psi_q^*(-\theta, \lambda), \tag{4.3.1}$$

where

$$\begin{aligned} \Psi_q(\theta, \lambda) &= \int_0^\infty q e^{-qt}\mathbb{E}\{e^{i\theta S_t - \lambda G_t}\} \\ &= \exp\left(\int_0^\infty dt \int_0^\infty e^{-qt}(e^{-\lambda t + i\theta x} - 1)t^{-1}\mathbb{P}\{X_t \in dx\}\right), \end{aligned} \tag{4.3.2}$$

and Ψ_q^* denotes the analogous quantity for X^*.

In a seminal paper Greenwood and Pitman [50] (see also [51]) reformulated the analytic identity (4.3.1) probabilistically and gave a proof of (4.3.2) using excursion theory. With $\boldsymbol{e} = \boldsymbol{e}_q$ being a random variable with an Exp(q) distribution which is independent of X, they wrote it in the form

$$(\boldsymbol{e}, X_{\boldsymbol{e}}) \overset{(d)}{=} (G_{\boldsymbol{e}}, S_{\boldsymbol{e}}) + (G^*_{\boldsymbol{e}}, -S^*_{\boldsymbol{e}}), \tag{4.3.3}$$

where the terms on the right are independent. This identity can be understood as follows. Duality, in other words time-reversal, shows that

$$(\boldsymbol{e} - G_{\boldsymbol{e}}, X_{\boldsymbol{e}} - S_{\boldsymbol{e}}) \overset{(d)}{=} (G^*_{\boldsymbol{e}}, -S^*_{\boldsymbol{e}}),$$

and since $(\boldsymbol{e} - G_{\boldsymbol{e}}, X_{\boldsymbol{e}} - S_{\boldsymbol{e}})$ is determined by the excursion away from 0 of R which straddles the exponentially distributed time $\boldsymbol{e}$, excursion theory makes the independence clear.

In Section VI.2 of [12] these results are established in a different way. The key points in that proof are:

- a proof of the independence referred to above by a direct argument;
- the fact that $(\boldsymbol{e}, X_{\boldsymbol{e}})$ has a bivariate infinitely divisible law with Lévy measure $t^{-1}e^{-qt}\mathbb{P}(X_t \in dx)dt$, $t > 0$, $x \in \mathbb{R}$;
- the fact that each of $(G_{\boldsymbol{e}}, S_{\boldsymbol{e}}), (\boldsymbol{e} - G_{\boldsymbol{e}}, X_{\boldsymbol{e}} - S_{\boldsymbol{e}})$ has a bivariate infinitely divisible law; write μ, μ^* for their Lévy measures;
- the conclusion that

$$\begin{aligned} \mu(dt, dx) &= t^{-1}e^{-qt}\mathbb{P}(X_t \in dx)dt,\ t > 0,\ x > 0, \\ \mu^*(dt, dx) &= t^{-1}e^{-qt}\mathbb{P}(X_t \in dx)dt,\ t > 0,\ x < 0. \end{aligned}$$

Then formula (4.3.2) follows from the Lévy–Khintchine formula, and (4.3.1) follows by using (4.3.2), the analogous result for $-X$, and the Frullani integral.

One of the few examples where the factorisation (4.3.1) is completely explicit is when X is Brownian motion; then it reduces to

$$\frac{q}{q+\lambda+\frac{1}{2}\theta^2} = \frac{\sqrt{2q}}{\sqrt{2(q+\lambda)}-i\theta} \cdot \frac{\sqrt{2q}}{\sqrt{2(q+\lambda)}+i\theta}.$$

Other cases where semi-explicit versions are available include the spectrally one-sided case, which we will discuss in detail in Chapter 9, and certain stable processes: see Doney [30].

If we remove the dependence on time by setting $\lambda = 0$, we get the "spatial Wiener–Hopf factorisation":

$$\mathbb{E}\left(e^{i\theta X_{\boldsymbol{e}}}\right) = \frac{q}{q+\Psi(\theta)} = \Psi_q(\theta,0)\Psi_q^*(-\theta,0) = \mathbb{E}\left(e^{i\theta S_{\boldsymbol{e}}}\right)\mathbb{E}\left(e^{i\theta(X_{\boldsymbol{e}}-S_{\boldsymbol{e}})}\right), \tag{4.3.4}$$

and the corresponding temporal result is

$$\mathbb{E}\left(e^{-\lambda \boldsymbol{e}}\right) = \frac{q}{q+\lambda} = \Psi_q(0,\lambda)\Psi_q^*(0,\lambda) = \mathbb{E}\left(e^{-\lambda G_{\boldsymbol{e}}}\right)\mathbb{E}\left(e^{-\lambda(\boldsymbol{e}-G_{\boldsymbol{e}})}\right). \tag{4.3.5}$$

However most applications of Wiener–Hopf factorisation are based on the following consequence of (4.3.2), which is due to Fristedt [48].

Theorem 10. *(Fristedt's formula) The exponent of the bivariate increasing ladder process is given for $\alpha, \beta \geq 0$, by*

$$\kappa(\alpha,\beta) = c\exp\left(\int_0^\infty \int_0^\infty \left(e^{-t} - e^{-\alpha t-\beta x}\right) t^{-1}\mathbb{P}\left\{X_t \in dx\right\} dt\right), \tag{4.3.6}$$

where c is a positive constant whose value depends on the normalization of the local time L.

Proof. (From [12], Section VI.2.) The crucial point is that we can show that

$$\mathbb{E}\left\{e^{-(\alpha G_{\boldsymbol{e}}+\beta S_{\boldsymbol{e}})}\right\} = \Psi_q(i\beta,\alpha) = \frac{\kappa(q,0)}{\kappa(q+\alpha,\beta)}, \tag{4.3.7}$$

and then (4.3.6) follows by comparing with the case $q = 1$ of (4.3.2). An outline of the proof of (4.3.7) in the case that 0 is regular for R follows. Because of this regularity X cannot make a positive jump at time $G_{\boldsymbol{e}}$ so we have $S_{\boldsymbol{e}} = S_{G_{\boldsymbol{e}}-}$ a.s. and

$$\mathbb{E}\{e^{-(\alpha G_{\boldsymbol{e}}+\beta S_{G_{\boldsymbol{e}}-})}\} = q\mathbb{E}\left[\int_0^\infty e^{-qt}e^{-(\alpha G_t+\beta S_{G_t-})}dt\right]$$

$$= q\mathbb{E}\left[\int_0^\infty e^{-qt}\mathbf{1}_{\{R_t=0\}}e^{-(\alpha t+\beta S_t-)}dt\right] + \mathbb{E}\left[\sum_g e^{-(\alpha g+\beta S_{g-})}\int_g^d qe^{-qt}dt\right],$$

where $\sum_g$ means summation over all the excursion intervals (g, d) of R. The first term above is 0 unless the inverse local time τ has a positive drift η, in which case, making the change of variable $t = \tau_u$, we see that it equals

$$\begin{aligned} & q\eta\mathbb{E}\left[\int_0^\infty e^{-qt} e^{-\{\alpha t + \beta S_t)} dL(t)\right] \\ &= q\eta\mathbb{E}\left[\int_0^\infty e^{-\{(\alpha+q)\tau_u + \beta H_u)} du\right] = \frac{q\eta}{\kappa(q+\alpha,\beta)}. \end{aligned}$$

Noting that

$$\int_g^d q e^{-qt} dt = e^{-qg} \int_0^{d-g} q e^{-qt} dt = e^{-qg}(1 - e^{-q\zeta}),$$

we can use the compensation formula to write the second term as

$$\mathbb{E}\left[\int_0^\infty e^{-(q+\alpha)t - \beta S_t} dL(t)\right] n(1 - e^{-q\zeta}) = \frac{n(1 - e^{-q\zeta})}{\kappa(q+\alpha,\beta)},$$

where n is the excursion measure of R and $\zeta = d - g$ the lifetime of the generic excursion. (Note we are using the standard abbreviation $n(f)$ for $\int_{\mathcal{E}} f(\varepsilon) n(d\varepsilon)$.) Since we know that the Laplace exponent of τ is given by $\kappa(q,0) = \eta q + n(1 - e^{-q\zeta})$, (4.3.7) follows, and hence the result. ■

4.4 Applications

We now discuss some straight-forward applications of the various Wiener–Hopf identities.

Corollary 3. *For some constant $c' > 0$ and all $\lambda > 0$,*

(i) the Laplace exponents of τ and τ^* satisfy

$$\kappa(\lambda,0)\kappa^*(\lambda,0) = c'\lambda; \tag{4.4.1}$$

(ii) the Laplace exponents of H and H^* satisfy

$$\kappa(0,-i\lambda)\kappa^*(0,i\lambda) = c'\Psi(\lambda). \tag{4.4.2}$$

Proof. (i) Applying Fristedt's formula to $-X$ we get a similar expession for $\kappa^*(\alpha,\beta)$, the exponent of the downgoing ladder process, which yields

$$\begin{aligned} \kappa(\lambda,0)\kappa^*(\lambda,0) &= c\hat{c}\exp\left(\int_0^\infty (e^{-t} - e^{-\lambda t}) t^{-1} \mathbb{P}\{X_t > 0\} dt\right) \\ &\quad \times \exp\left(\int_0^\infty (e^{-t} - e^{-\lambda t}) t^{-1} \mathbb{P}\{X_t < 0\} dt\right) \\ &= c' \exp\left(\int_0^\infty (e^{-t} - e^{-\lambda t}) t^{-1} dt\right) = c'\lambda, \end{aligned}$$

where we have again used the Frullani integral.

(ii) Comparing (4.3.4) with (4.3.7) we see that the Wiener–Hopf factors satisfy

$$\Psi_q(\lambda,0)=\frac{\kappa(q,0)}{\kappa(q,-i\lambda)},\ \Psi_q^*(\lambda,0)=\frac{\kappa^*(q,0)}{\kappa^*(q,i\lambda)}.$$

Using (4.4.1) and (4.3.4) gives

$$\begin{aligned}\frac{1}{\Psi(\lambda)}&=\lim_{q\downarrow 0}\frac{1}{q+\Psi(\lambda)}=\lim_{q\downarrow 0}\frac{1}{q}\frac{\kappa(q,0)\kappa^*(q,0)}{\kappa(q,-i\lambda)\kappa^*(q,i\lambda)}\\&=\frac{c'}{\kappa(0,-i\lambda)\kappa^*(0,i\lambda)}.\end{aligned}$$

■

The relation (4.4.2) is often referred to as the Wiener–Hopf factorisation of the Lévy exponent, and corresponds to the Wiener–Hopf factorisation of the characteristic function in random-walk theory, (4.2.3). It has some important consequences, the first of which follow.

Corollary 4. *(i) The drifts δ and δ^* of H and H^* satisfy*

$$2\delta\delta^*=\sigma^2;$$

(ii) If $\mathbb{E}|X_1|<\infty$ and $\mathbb{E}X_1=0$ the means $m=\mathbb{E}H_1$ and $m^*=\mathbb{E}H_1^*$ satisfy

$$2mm^*=VarX_1=\sigma^2+\int_{-\infty}^{\infty}x^2\Pi(dx)\le\infty.$$

(iii) At most one of $H,H^*(\tau,\tau^*)$ has a finite lifetime, and $H(\tau)$ has a finite lifetime if and only if $\int_1^\infty t^{-1}\mathbb{P}(X_t\ge 0)dt<\infty$. This happens if and only if $X_t\to-\infty$ a.s. as $t\to\infty$.

(iv) If $X_t\to\infty$ a.s. as $t\to\infty$ then

$$\mathbb{E}(X_1)=\kappa^*(0,0)\mathbb{E}(H_1)=k^*\mathbb{E}(H_1)\le\infty.$$

(v) If X is not a compound Poisson process then at most one of H, $H^*(\tau,\tau^*)$ is a compound Poisson process and $H(\tau)$ is a compound Poisson process if and only if $\int_0^1 t^{-1}\mathbb{P}(X_t\ge 0)dt<\infty$. This happens if and only if τ^* has a positive drift.

Proof. (i) This follows by dividing (4.4.2) by λ^2 and letting $\lambda\to\infty$, and (ii) is the same, but letting $\lambda\downarrow 0$. For (iii) observe that Fristedt's formula gives

$$\begin{aligned}\lim_{\beta\downarrow 0}\kappa(0,\beta)&>0\Leftrightarrow\lim_{\alpha\downarrow 0}\kappa(\alpha,0)>0\\&\Longleftrightarrow\int_0^\infty t^{-1}\left(1-e^{-t}\right)\mathbb{P}(X_t>0)dt<\infty\\&\Longleftrightarrow\int_1^\infty t^{-1}\mathbb{P}\left(X_t>0\right)dt<\infty,\end{aligned}$$

and of course $\int_1^\infty t^{-1}\mathbb{P}(X_t > 0)dt + \int_1^\infty t^{-1}\mathbb{P}(X_t < 0)dt = \infty$. (iv) follows by dividing (4.4.2) by λ and letting $\lambda \to 0$. For (v) note that

$$\begin{aligned}\lim_{\beta\to\infty}\kappa(0,\beta) &< \infty \Leftrightarrow \lim_{\alpha\to\infty}\kappa(\alpha,0) < \infty \\ &\Longleftrightarrow \int_0^\infty t^{-1}e^{-t}\mathbb{P}(X_t > 0) < \infty \\ &\Longleftrightarrow \int_0^1 t^{-1}\mathbb{P}(X_t > 0) < \infty.\end{aligned}$$

The final statement then follows by letting $\lambda \to \infty$ in (4.4.1). ■

It is clear that H is a compound Poisson process if and only if $(0,\infty)$ is irregular for X, so this result implies that either both half-lines are regular, or exactly one is. Similarly either exactly one of the ladder processes has infinite lifetime or both have; this corresponds to the trichotomy, familiar from random walks, of oscillation, drift to ∞, or drift to $-\infty$. The integral tests given above are originally due to Rogozin [88]; note they are not expressed directly in terms of the characteristics of X.

Specialising Fristedt's formula gives an expression for the exponent $\kappa(\lambda,0)$ of τ which is usually ascribed to Spitzer; with $\rho(t) = P(X_t > 0)$ it is

$$\kappa(\lambda,0) = c\exp\left(\int_0^\infty (e^{-t} - e^{-\lambda t})t^{-1}\rho(t)dt\right), \quad \lambda \geq 0. \tag{4.4.3}$$

Since $\kappa(\lambda,0)$ determines the distribution of the ladder time process τ, we see that the quantity $\rho(t)$ is just as important in the study of Lévy processes as the corresponding quantity is for random walks. For example, the continuous-time version of Spitzer's condition,

$$\frac{1}{t}\int_0^t \rho(s)ds \to \rho \in (0,1) \text{ as } t \to \infty, \text{ (respectively } t \downarrow 0), \tag{4.4.4}$$

is equivalent to $\kappa(\lambda,0) \in RV(\rho)$ as $\lambda \downarrow 0$, (respectively $\lambda \to \infty$), and this happens if and only if τ belongs to the domain of attraction of a ρ-stable process as $t \to \infty$, (respectively $t \downarrow 0$). Since G_t coincides with $\tau(T_t-)$, where T is the first passage process of τ, it is not surprising that (4.4.4) is also the necessary and sufficient condition for $t^{-1}G_t \xrightarrow{D}$ generalised arc-sine law of parameter ρ as $t \to \infty$, (respectively $t \downarrow 0$). This also extends to the cases $\rho = 0, 1$, the corresponding limit being a unit mass at 0 or 1. For details see Theorem 14, p. 169 of [12].

The more familiar form of the arc-sine theorem involves not G_t, but rather the quantity $A_t = \int_0^t 1_{\{X_s>0\}}ds$. However, just as for random walks, the "Sparre Andersen Identity",

$$A_t \overset{(d)}{=} G_t, \tag{4.4.5}$$

holds for each $t > 0$, so the same assertion holds with G_t replaced by A_t. Note that whereas the random-walk version of (4.4.5) can be established by a combinatorial argument due to Feller, this doesn't seem possible in the Lévy process case.

Next we introduce the renewal function U associated with H, which is given by

$$U(x) = \int_0^\infty \mathbb{P}(H_t \le x)dt = \mathbb{E}\left(\int_0^\infty \mathbf{1}_{(S_t \le x)} dL(t)\right), \ 0 \le x < \infty, \qquad (4.4.6)$$

so that

$$\lambda \int_0^\infty e^{-\lambda x} U(x)dx = \frac{1}{\kappa(0,\lambda)}, \ \lambda > 0.$$

This quantity is closely related to $T_x = T_{(x,\infty)}$, as the following shows.

Proposition 6. *(i) If X drifts to $-\infty$, then for some $c > 0$ and all $x \ge 0$*

$$U(x) = c\mathbb{P}(S_\infty \le x) = c\mathbb{P}(T_x = \infty).$$

(ii) If X drifts to ∞, then for some $c > 0$ and all $x \ge 0$

$$U(x) = c\mathbb{E}(T_x) < \infty.$$

(iii) If X oscillates, then $\mathbb{P}(S_\infty < \infty) = 0$ and for each $x > 0$

$$\mathbb{E}(T_x) = \infty.$$

(iv) Spitzer's condition (4.4.4) holds with $0 < \rho < 1$ if and only if for some, and then all, $x > 0$, $\mathbb{P}(T_x > \cdot) \in RV(-\rho)$ at ∞, and when this happens

$$\lim_{t\to\infty} \frac{\mathbb{P}(T_x > t)}{\mathbb{P}(T_y > t)} = \frac{U(x)}{U(y)} \text{ for every } x, y > 0.$$

Proof. We will just indicate the proof of (iv). Specializing (4.3.7) we see that

$$\begin{aligned}\frac{\kappa(q,0)}{\kappa(q,\lambda)} &= \mathbb{E}\left(e^{-\lambda S_{e_q}}\right) = \int_0^\infty e^{-\lambda x}\mathbb{P}(S_{\boldsymbol{e}_q} \in dx) \\ &= \lambda \int_0^\infty e^{-\lambda x}\mathbb{P}(S_{\boldsymbol{e}_q} \le x)dx = \lambda \int_0^\infty e^{-\lambda x}\mathbb{P}\left(T_x > \boldsymbol{e}_q\right)dx,\end{aligned}$$

which we can invert to get

$$1 - \mathbb{E}\left(e^{-qT_x}\right) = q\int_0^\infty e^{-qt}\mathbb{P}(T_x > t)dt = \kappa(q,0)U^{(q)}(x), \qquad (4.4.7)$$

where

$$U^{(q)}(x) = \int_0^\infty \mathbb{E}(e^{-q\tau_t}; H_t \le x)dt = \mathbb{E}\left(\int_0^\infty e^{-qu}\mathbf{1}_{(S_u \le x)}dL(u)\right)$$

satisfies

$$\lambda \int_0^\infty e^{-\lambda x} U^{(q)}(x)dx = \frac{1}{\kappa(q,\lambda)}, \ \lambda > 0. \tag{4.4.8}$$

Since clearly $U^{(q)}(x) \uparrow U(x)$ as $q \to 0$ for each $x > 0$, the result follows from (4.4.7) by standard Tauberian arguments. ■

We will finish this section with another result from [12] involving the passage time T_x. It is the Lévy process version of a result that was proved for random walks by Spitzer; see P3, p. 209 in [94]. It is interesting to see how we need a fair amount of machinery to extend this simple result to the continuous time setting.

Theorem 11. *(Bertoin). Assume X is not a compound Poisson process. Then for $x, u > 0$,*

$$\mathbb{P}(X_{T_x} \in x + du) = c' \int_{y=0}^{x} \int_{v \geq x-y} U(dy)U^*(dv + y - x)\Pi(v + du); \tag{4.4.9}$$

Proof. It is enough to prove that, for a.e. $v \geq 0$,

$$\int_{t=0}^{\infty} \mathbb{P}(X_t \in x - dv, t < T_x)dt = c' \int_{(x-v)^+}^{x} U(dy)U^*(dv + y - x),$$

since (4.4.9) then follows by the compensation formula. To do this we use (4.3.4), which we can restate as $X_{\boldsymbol{e}_q} \overset{(d)}{=} S_{\boldsymbol{e}_q} - \widetilde{S}^*_{\boldsymbol{e}_q}$, where $\widetilde{S}^*_{\boldsymbol{e}_q}$ is an independent copy of $S^*_{\boldsymbol{e}_q}$. Note that

$$\begin{aligned}
\int_{t=0}^{\infty} \mathbb{P}(X_t \in dw, t < T_x)dt &= \lim_{q \to 0} \int_{t=0}^{\infty} e^{-qt} \mathbb{P}(X_t \in dw, t < T_x)dt \\
&= \lim_{q \to 0} q^{-1} \mathbb{P}\left(X_{\boldsymbol{e}_q} \in dw, \boldsymbol{e}_q < T_x\right) \\
&= \lim_{q \to 0} q^{-1} \mathbb{P}\left(S_{\boldsymbol{e}_q} \leq x, S_{\boldsymbol{e}_q} - \widetilde{S}^*_{\boldsymbol{e}_q} \in dw\right) \\
&= \lim_{q \to 0} q^{-1} \int_{w^+}^{x} \mathbb{P}(S_{\boldsymbol{e}_q} \in dy) \mathbb{P}(S^*_{\boldsymbol{e}_q} \in y - dw) \\
&= c' \lim_{q \to 0} \int_{w^+}^{x} \frac{\mathbb{P}(S_{\boldsymbol{e}_q} \in dy)}{\kappa(q,0)} \frac{\mathbb{P}(S^*_{\boldsymbol{e}_q} \in y - dw)}{\kappa^*(q,0)},
\end{aligned}$$

where we have used (4.4.1) in the last step. But using (4.3.7) we see that as $q \to 0$,

$$\frac{\mathbb{E}(e^{-\lambda S_{\boldsymbol{e}_q}})}{\kappa(q,0)} = \frac{1}{\kappa(q,\lambda)} \to \frac{1}{\kappa(0,\lambda)} = \int_0^\infty e^{-\lambda x} U(dx),$$

which gives the weak convergence of the first term in the integral to $U(dy)$, and since the same argument applies to the second part, the result follows. ■

The following important complement to this result deals with the possibility that the process passes continuously over the level x. The result is very natural once we observe that $\mathbb{P}(X_{T_x} = x)$ is the same as the probability that H creeps over the level x, but we omit the details of the proof, which is due to Millar in [76].

Theorem 12. *Assume X is not a compound Poisson process. Then for $x > 0$, $\mathbb{P}(X_{T_x} = x) \equiv 0$ unless the ladder height process has a drift $\delta_+ > 0$. In this case $U(dx)$ is absolutely continuous and there is a version u of its density which is bounded, continuous and positive on $(0, \infty)$ and has $\lim_{x\downarrow 0} u(x) = u(0+) > 0$; moreover*

$$\mathbb{P}(X_{T_x} = x) = \frac{u(x)}{u(0+)}.$$

4.5 A Stochastic Bound

In this section we show how the independence between $S_{\boldsymbol{e}_q}$ and $X_{\boldsymbol{e}_q} - S_{\boldsymbol{e}_q}$ leads to a useful stochastic bound for the sample paths of X in terms of random walks.

We would frequently like to be able to assert that some aspect of the behaviour of X as $t \to \infty$ can be seen to be true "by analogy with known results for random walks". An obvious way to try to justify such a claim is via the random walk $S^{(\delta)} := (X(n\delta), n \geq 0)$, for fixed $\delta > 0$. (This process is often called the δ-skeleton of X.) However it can be difficult to control the deviation of X from $S^{(\delta)}$. A further problem stems from the fact that the distribution of $S_1^{(\delta)} = X(\delta)$ is determined via the Lévy–Khintchine formula and not directly in terms of the characteristics of X.

An alternative approach is to use the random walk which results from observing X at the times at which its "large jumps" occur. Specifically we assume that $\Pi(\mathbb{R}) > 0$, and take a fixed interval $I = [-\eta, \eta]$ which contains zero and has $\Delta := \Pi(I^c) > 0$, put $\tau_0 = 0$, and for $n \geq 1$ write τ_n for the time at which J_n, the nth jump in X whose value lies in I^c, occurs. The random walk is then defined by

$$\hat{S} := (\hat{S}_n, n \geq 0), \text{ where } \hat{S}_n = X(\tau_n). \tag{4.5.1}$$

Of course $(\tau_n, n \geq 1)$ is the sequence of arrival times in a Poisson process of rate Δ which is independent of $(J_n, n \geq 1)$, and this latter is a sequence of independent, identically distributed random variables having the distribution $\Delta^{-1}\mathbf{1}_{I^c}\Pi(dx)$. We will write $\hat{Y}_1, \hat{Y}_2 \cdots$ for the steps in $\hat{S}$, so that with $e_r := \tau_r - \tau_{r-1}$, $\tau_0 = 0$, and $r \geq 1$

$$\hat{Y}_r = X(\tau_r) - X(\tau_{r-1}) = J_r + \widetilde{X}(\tau_r) - \widetilde{X}(\tau_{r-1}) \stackrel{D}{=} J_r + \widetilde{X}(e_r), \tag{4.5.2}$$

where $\widetilde{X}$ is "X with the jumps $J_1, J_2, \cdots$ removed". This is also a Lévy process whose Lévy measure is the restriction of Π to I. Furthermore $\widetilde{X}$ is independent of $\{(J_n, \tau_n), n \geq 1\}$, and since it has no large jumps, it follows that $E\{e^{\lambda \widetilde{X}_t}\}$

is finite for all real λ. Thus the contribution of $\sum_1^n \widetilde{X}(e_r)$ to $\hat{S}_n$ can be easily estimated, and for many purposes $\hat{Y}_r$ can be replaced by $J_r + \widetilde{\mu}$, where $\widetilde{\mu} = E\widetilde{X}(\tau_1)$. In order to control the deviation of X from $\hat{S}$ it is natural to use the stochastic bounds

$$I_n \leq X_t \leq M_n \text{ for } \tau_n \leq t < \tau_{n+1}, \tag{4.5.3}$$

where

$$I_n := \inf_{\tau_n \leq t < \tau_{n+1}} X_t, \; M_n := \sup_{\tau_n \leq t < \tau_{n+1}} X_t, \tag{4.5.4}$$

and write

$$M_n = \hat{S}_n + \widetilde{m}_n, \text{ and } I_n = \hat{S}_n + \widetilde{\imath}_n. \tag{4.5.5}$$

Here

$$\widetilde{m}_n = \sup_{0 \leq s < e_{n+1}} \left\{ \widetilde{X}(\tau_n + s) - \widetilde{X}(\tau_n) \right\}, \; n \geqslant 0, \tag{4.5.6}$$

$$\widetilde{\imath}_n = \inf_{0 \leq s < e_{n+1}} \left\{ \widetilde{X}(\tau_n + s) - \widetilde{X}(\tau_n) \right\}, \; n \geqslant 0, \tag{4.5.7}$$

are each independent identically distributed sequences, and both $\widetilde{m}_n$ and $\widetilde{\imath}_n$ are independent of $\hat{S}_n$. This method also leads to some technical complications; see for example the proofs of Theorems 3.3 and 3.4 in [40].

But there is a different way to represent the random variables M_n and I_n in (4.5.4).

Theorem 13. *Using the above notation we have, for any fixed* $\eta > 0$ *with* $\Delta = \Pi(I^c) > 0$,

$$M_n = S_n^{(+)} + \widetilde{m}_0, \; I_n = S_n^{(-)} + \widetilde{\imath}_0, \; n \geq 0, \tag{4.5.8}$$

where each of the processes $S^{(+)} = (S_n^{(+)}, n \geq 0)$ *and* $S^{(-)} = (S_n^{(-)}, n \geq 0)$ *are random walks with the same distribution as* $\hat{S}$. *Moreover* $S^{(+)}$ *and* $\widetilde{m}_0$ *are independent, as are* $S^{(-)}$ *and* $\widetilde{\imath}_0$.

Comparing the representations (4.5.5) and (4.5.8), note that for each fixed n the pairs $(\hat{S}_n, \widetilde{m}_n)$ and $(S_n^{(+)}, \widetilde{m}_0)$ have the same joint law; however the latter representation has the great advantage that the term $\widetilde{m}_0$ does not depend on n.

Proof of Theorem 13. The Wiener–Hopf factorisation (4.3.4) for $\widetilde{X}$ asserts that the random variables $\widetilde{m}_0 = \sup_{0 \leq t < e_1} \widetilde{X}_t$ and $\widetilde{X}_{e_1} - \widetilde{m}_0$ are independent, and that the latter has the same distribution as $\widetilde{\imath}_0 = \inf_{0 \leq t < e_1} \widetilde{X}_t$. (Recall that $\widetilde{X}$ and e_1 are independent and e_1 has an $Exp(\Delta)$ distribution.) Since

$$\begin{aligned} M_1 &= \sup_{e_1 \leq t < e_1 + e_2} X_t = \widetilde{X}(e_1) + J_1 + \sup_{0 \leq t < e_2} \left\{ \widetilde{X}(e_1 + t) - \widetilde{X}(e_1) \right\} \\ &= \widetilde{m}_0 + \left\{ \widetilde{X}(e_1) - \widetilde{m}_0 \right\} + J_1 + \widetilde{m}_1 \\ &:= \widetilde{m}_0 + Y_1^{(+)}, \end{aligned}$$

where all four random variables in the second line are independent, we see that $Y_1^{(+)}$ is independent of $\widetilde{m}_0$ and has the same distribution as $J_1 + \widetilde{X}(e_1)$, and hence as $X(e_1)$. A similar calculation applied to M_n gives the required conclusions for $S^{(+)}$, and since $S^{(-)}$ is $S^{(+)}$ evaluated for $-X$, the proof is finished. ■

A straightforward consequence of Theorem 13 is

Proposition 7. *Suppose that $b \in RV(\alpha)$, and $\alpha > 0$. Then for any fixed $\eta > 0$ with $\Delta = \Pi(I^c) > 0$, and any $c \in [-\infty, \infty]$*

$$\frac{\hat{S}_n}{b(n)} \stackrel{a.s.}{\to} c \text{ as } n \to \infty \Longleftrightarrow \frac{X_t}{b(t)} \stackrel{a.s.}{\to} c\Delta^\alpha \text{ as } t \to \infty. \tag{4.5.9}$$

(Here $RV(\alpha)$ denotes the class of positive functions which are regularly varying with index α at ∞.)

Proof of Proposition 7. With $N_t = \max\{n : \tau_n \leq t\}$ we have, from (4.5.3) and (4.5.8),

$$\frac{\widetilde{\imath}_0}{b(t)} + \frac{S_{N_t}^{(-)}}{b(N_t)} \cdot \frac{b(N_t)}{b(t)} \leq \frac{X_t}{b(t)} \leq \frac{S_{N_t}^{(+)}}{b(N_t)} \cdot \frac{b(N_t)}{b(t)} + \frac{\widetilde{m}_0}{b(t)}. \tag{4.5.10}$$

Clearly the extreme terms converge a.s. to zero as $t \to \infty$, and by the strong law $b(N_t)/b(t) \stackrel{a.s.}{\to} \Delta^\alpha$. So if $\frac{\hat{S}_n}{b(n)} \stackrel{a.s.}{\to} c$ as $n \to \infty$, then $\frac{S_n^{(+)}}{b(n)} \stackrel{a.s.}{\to} c$ and $\frac{S_n^{(-)}}{b(n)} \stackrel{a.s.}{\to} c$, and hence $\frac{X_t}{b(t)} \stackrel{a.s.}{\to} c\Delta^\alpha$ as $t \to \infty$. On the other hand, if this last is true we can use (4.5.10) with $t = \tau_n$ to reverse the argument. ■

From this, and the analogous statements which hold for limsup and liminf, known results about Lévy processes such as strong laws and laws of the iterated logarithm can easily be deduced. But there is a vast literature on the asymptotic behaviour of random walks, and by no means all the results it contains have been extended to the setting of Lévy processes. Using Theorem 13 we can show, for example, that the classical results of Kesten in [59] about strong limit points of random walks, and results about the limsup behaviour of S_n/n^α and $|S_n|/n^\alpha$ and hence about first passage times outside power-law type boundaries in [63], all carry over easily: see [43]. In [36] this method was used to extend to the Lévy-process setting the extensive results by Kesten and Maller in [60], [62], and [64], about various aspects of the asymptotic behaviour of random walks which converge to $+\infty$ in probability. Results about existence of moments for first and last passage times in the transient case from [57] and [61] were similarly extended in [42]. Here we will illustrate the method by giving a new proof of an old result for random walks due to Erickson [46], and then giving the Lévy-process version.

Suppose that $(S_n, n \geq 0)$ is a random walk with $S_0 = 0$ and $S_n = \sum_1^n Y_r$ for $n \geq 1$, where the Y_r are independent and identically distributed copies of a random variable Y which has

$$EY^+ = EY^- = \infty. \tag{4.5.11}$$

Lemma 3. *Write* $\overline{S}_n = \max_{r \le n} S_r$, *and assume (4.5.11) and* $S_n \stackrel{a.s.}{\to} \infty$. *Then*

$$\frac{S_n}{\overline{S}_n} \stackrel{a.s.}{\to} 1. \tag{4.5.12}$$

Proof. As in Section 2 we write T_n and H_n for the time and position of the nth strict increasing ladder event, with $T_0 = H_0 \equiv 0$, and for $k \ge 1$ let

$$D_k = \max_{T_{k-1} \le j \le T_k} \{H_{k-1} - S_j\} \tag{4.5.13}$$

denote the depth of the kth excursion below the maximum. Note that the D_k are independent and identically distributed and

$$1 - \frac{S_n}{\overline{S}_n} = \frac{\overline{S}_n - S_n}{\overline{S}_n} \le \frac{D_{N_n+1}}{H_{N_n}}, \tag{4.5.14}$$

where $N_n = \max\{k : T_k \le n\}$ is the number of such excursions completed by time n. Since $N_n \stackrel{a.s.}{\to} \infty$ it is clear that (4.5.12) will follow if we can show

$$\frac{D_{n+1}}{H_n} \stackrel{a.s.}{\to} 0, \tag{4.5.15}$$

and this in turn will follow if we can show that for every $\varepsilon > 0$

$$\sum_{n=0}^{\infty} P\{D_{n+1} > \varepsilon H_n\} < \infty. \tag{4.5.16}$$

However, since D_{n+1} is independent of H_n, this in turn will follow from $EV(\varepsilon^{-1}D_1) < \infty$, where $V(y)$ is the renewal function $\sum_0^\infty P(H_k \le y)$. Now as V is subadditive it is easy to see that this sum either converges for all $\varepsilon > 0$ or diverges for all $\varepsilon > 0$. But since $D_{n+1} > H_n$ occurs if and only if the random walk visits $(-\infty, 0)$ during the nth excursion below the maximum, when $\varepsilon = 1$ the sum of the series in (4.5.16) is $E_0 N$, where N is the total number of excursions with this property. A moment's thought shows that

$$E_0 N = p\,(1 + E(E_M N)) \le p(1 + E_0 N),$$

where $p := P(N > 0) < 1$, and M denotes the position of the random walk at the end of the first excursion that visits $(-\infty, 0)$, so the result follows. ■

Corollary 5. *Whenever* $S_n \stackrel{a.s.}{\to} \infty$ *and (4.5.11) holds we have* $n^{-1}S_n \stackrel{a.s.}{\to} \infty$.

Proof. By Lemma 3 we need only prove that $n^{-1}\overline{S}_n \stackrel{a.s.}{\to} \infty$. But a consequence of drift to ∞ is that $ET_1 < \infty$, and, because $P(H_1 > x) \ge P(Y_1 > x)$, a consequence of (4.5.11) is that $EH_1 = \infty$. Writing

$$\frac{\overline{S}_n}{n} = \frac{\sum_1^{N_n}\{H_r - H_{r-1}\}}{N_n} \cdot \frac{N_n}{n},$$

we see that the result follows by the strong law, as on the right-hand side the first term $\stackrel{a.s.}{\to} \infty$ and the second term $\stackrel{a.s.}{\to} 1/ET_1$. ■

The result we are aiming at follows.

Theorem 14. *(Erickson) Assume (4.5.11) holds and write*

$$B^{\pm}(x) = \frac{x}{A^{\pm}(x)} \text{ where } A^{\pm}(x) = \int_0^x P(Y^{\pm} > y)dy.$$

Then one of the following alternatives must hold;

(i) $S_n \overset{a.s.}{\to} \infty, n^{-1}S_n \overset{a.s.}{\to} \infty$, *and* $EB^+(Y^-) < \infty$;
(ii) $S_n \overset{a.s.}{\to} -\infty, n^{-1}S_n \overset{a.s.}{\to} -\infty$, *and* $EB^-(Y^+) < \infty$;
(iii) S_n *oscillates,* $\liminf n^{-1}S_n \overset{a.s.}{=} -\infty$, $\limsup n^{-1}S_n \overset{a.s.}{=} \infty$, *and* $EB^+(Y^-) = EB^-(Y^+) = \infty$.

Proof. First we remark that Corollary 5 implies that for any fixed $K > 0$, $n^{-1}\{S_n - Kn\} \overset{a.s.}{\to} \infty$ if $S_n \overset{a.s.}{\to} \infty$, and $n^{-1}\{S_n + Kn\} \overset{a.s.}{\to} -\infty$ if $S_n \overset{a.s.}{\to} -\infty$. This then implies that if S_n oscillates, both of the walks $S_n \pm Kn$ also oscillate, and hence

$$\limsup n^{-1}S_n = \limsup n^{-1}\{S_n - nK\} + K \overset{a.s.}{\geq} K,$$
$$\liminf n^{-1}S_n = \liminf n^{-1}\{S_n + nK\} - K \overset{a.s.}{\leq} -K.$$

Since K is arbitrary, this means that $\liminf n^{-1}S_n \overset{a.s.}{=} -\infty$, and $\limsup n^{-1}S_n \overset{a.s.}{=} \infty$. The same argument shows that if $\{\widetilde{S}_n, n \geq 0\}$ is any random walk with the property that, for some finite K and all $n \geq 1$,

$$|\widetilde{S}_n - S_n| \leq nK,$$

then either both walks drift to ∞, both drift to $-\infty$, or both oscillate.

Suppose now that S_n either drifts to $-\infty$ or oscillates, so that the first weak downgoing ladder height H_1^- is proper. Then, integrating (4.2.6) applied to $-S$, gives

$$1 = P\left(H_1^- \in (-\infty, 0]\right) = \int_0^\infty P\left(Y^- \geq y\right) dV(y) \tag{4.5.17}$$
$$= \int_0^\infty V(y)P\left(Y^- \in dy\right) = E\left(V(Y^-)\right).$$

In view of the inequality $P(H_1 > y) \geq P(Y_1 > y)$ and the well-known "Erickson bound", valid for any renewal function,

$$1 \leq \frac{V(x)}{B^*(x)} \leq 2, \tag{4.5.18}$$

where $B^*(x) = x/A^*(x)$, and $A^*(x) = \int_0^x P(H > y)dy$ (see Lemma 1 of [46]), we see that $V(x) \leq 2B^*(x) \leq 2B^+(x)$, and hence, from (4.5.17),

$$EB^+(Y^-) \geq 1/2.$$

However this inequality is also valid for the random walk defined by $\widetilde{S}_n = S_n - \sum_1^n Y_r \mathbf{1}_{\{Y_r \in (-K,0]\}}$, which has $\widetilde{B}^+ = B^+$, so that

$$EB^+(Y^-; Y^- \geq K) \geq 1/2.$$

Since K is arbitrary we conclude that $EB^+(Y^-) = \infty$. We then see that always at least one of $EB^+(Y^-)$ and $EB^-(Y^+)$ is infinite and when S_n oscillates both are. Also the argument following (4.5.16) shows that when S_n drifts to ∞ we have $EV(D_1) < \infty$, which again by the Erickson bound means that $EB^*(D_1) < \infty$. Since $P(D_1 > x) > P(Y^- > x)$ it follows that $EB^*(Y^-) < \infty$. Finally we see that

$$\begin{aligned} P(H_1 > x) &= \sum P(T_1 > n, S_n + Y_{n+1} > x) \\ &\leq P(Y > x) \sum P(T_1 > n) = ET_1 P(Y > x), \end{aligned}$$

so that $B^*(x) \geq cB^+(x)$, and hence $EB^+(Y^-) < \infty$. ∎

The Lévy process version of this is:

Theorem 15. *Let X be any Lévy process with $\mathbb{E}X_1^+ = \mathbb{E}X_1^- = \infty$. Write Π^* for the Lévy measure of $-X$ and*

$$I^+ = \int_1^\infty \frac{x\Pi^*(dx)}{A(x)} \ , \ I^- = \int_1^\infty \frac{x\Pi(dx)}{A^*(x)}, \text{ where}$$
$$A(x) = \int_0^x \overline{\Pi}(y)dy, \text{ and } A^*(x) = \int_0^x \overline{\Pi^*}(y)dy, .$$

Then one of the following alternatives must hold;

(i) $X_t \overset{a.s.}{\to} \infty, t^{-1}X_t \overset{a.s.}{\to} \infty$ *as* $t \to \infty$, *and* $I^+ < \infty$;
(ii) $X_t \overset{a.s.}{\to} -\infty, t^{-1}X_t \overset{a.s.}{\to} -\infty$ *as* $t \to \infty$, *and* $I^- < \infty$;
(iii) X *oscillates,* $\liminf t^{-1}X_t \overset{a.s.}{=} -\infty, \limsup t^{-1}X_t \overset{a.s.}{=} \infty$, *and* $I^+ = I^- = \infty$.

Proof. Take any $\eta > 0$ with $\Delta = \Pi(I^c) > 0$ and note that $\hat{S}$ satisfies (4.5.11), and furthermore that I^+ (respectively I^-) is finite if and only if $E\hat{B}^+(\hat{Y}^-) < \infty$ (respectively $E\hat{B}^-(\hat{Y}^+) < \infty$). As previously mentioned, Proposition 7 is valid with lim replaced by lim inf or lim sup. The results then follow from Theorem 14. ∎

5

Further Wiener–Hopf Developments

5.1 Introduction

In the last ten years or so there have been several new developments in connection with the Wiener–Hopf equations for Lévy processes, and in this chapter I will describe some of them, and try to indicate how each of them is tailored to specific applications.

5.2 Extensions of a Result due to Baxter

We start by giving the Lévy process version of (4.2.3) from Chapter 4, which constitutes a direct connection between the law of the bivariate ladder process and the law of X, without intervention of transforms. We can deduce this from Fristedt's formula, but it is not difficult to see that this result also implies Fristedt's formula.

Proposition 8. *We have the following identity between measures on* $(0, \infty) \times (0, \infty)$*:*

$$\frac{1}{t}\mathbb{P}\{X_t \in dx\}dt = \int_0^\infty \mathbb{P}\{\tau(u) \in dt, H(u) \in dx\}\frac{du}{u}. \tag{5.2.1}$$

The proof in [18] works by showing that both sides have the same bivariate Laplace transform. We omit the details, as (5.2.1) is a special case of the next result.

We will see this result used in Chapter 7, and it has also been applied by Vigon in [101].

Note that integrating (5.2.1) gives the following:

$$\int_0^\infty \mathbb{P}\{X_t \in dx\}\frac{dt}{t} = \int_0^\infty \mathbb{P}\{H(u) \in dx\}\frac{du}{u}. \tag{5.2.2}$$

This states that the so-called "harmonic renewal measure" of X agrees with that of H on $(0,\infty)$. These objects have been studied in the random walk context in [49] and [38], and for Lévy processes in [84].

Is it possible to give a useful "disintegration" of (5.2.1)? This question was answered affirmatively for random walks in [4], and for Lévy processes in [1] and [2]. (See also [75] and [3] for further developments of these ideas.) Note that, in the standard notation, $T_x = \tau(H^{-1}(x))$, so that if we put $\sigma_x := L(T_x), x \ge 0$, then σ is the right-continuous inverse of H.

Proposition 9. *We have the following identity between measures on* $(0,\infty)^3$*:*

$$\frac{\mathbb{P}\{X_t \in dx, \sigma_x \in du\}dt}{t} = \frac{\mathbb{P}\{\tau(u) \in dt, H(u) \in dx\}du}{u}. \tag{5.2.3}$$

Proof. Note first that it suffices to prove that

$$\begin{aligned} I(dt,dx) &:= \int_0^v \mathbb{P}\left\{\tau(u) \in dt, H(u) \in dx\right\} \frac{du}{u} \\ &= \frac{\mathbb{P}\left\{X_t \in dx, \sigma_x \le v\right\} dt}{t} \\ &= \frac{\mathbb{P}\left\{X_t \in dx, H_v \ge x\right\} dt}{t} \\ &= \frac{\mathbb{P}\left\{X_t \in dx\right\} dt}{t} - \frac{\mathbb{P}\{X_t \in dx, H_v < x\}dt}{t}. \end{aligned} \tag{5.2.4}$$

On the one hand

$$\begin{aligned} \frac{d}{d\lambda}\int_0^\infty \int_0^\infty e^{-(\lambda t + \mu x)} I(dt,dx) &= \frac{d}{d\lambda}\int_0^v e^{-u\kappa(\lambda,\mu)}\frac{du}{u} \\ &= \frac{-\frac{d}{d\lambda}\kappa(\lambda,\mu)}{\kappa(\lambda,\mu)}\left(1 - e^{-v\kappa(\lambda,\mu)}\right). \end{aligned}$$

On the other hand

$$\begin{aligned} &-\lambda\frac{d}{d\lambda}\int_0^\infty \int_0^\infty e^{-(\lambda t+\mu x)} \frac{\mathbb{P}\left\{X_t \in dx, H_v < x\right\} dt}{t} \\ &= \lambda \int_0^\infty \int_0^\infty e^{-(\lambda t+\mu x)} \mathbb{P}\left\{X_t \in dx, H_v < x\right\} dt \\ &= \mathbb{E}\left\{e^{-\mu X_{\mathbf{e}_\lambda}}; H_v < X_{\mathbf{e}_\lambda}\right\} = \mathbb{E}\left\{e^{-\mu X_{\mathbf{e}_\lambda}}; H_v < X_{\mathbf{e}_\lambda}, \tau_v \le \mathbf{e}_\lambda\right\} \\ &= \mathbb{E}\left\{e^{-\mu(H_v + \widetilde{X}_{\widetilde{\mathbf{e}}_\lambda})}; \widetilde{X}_{\widetilde{\mathbf{e}}_\lambda} > 0, \tau_v \le \mathbf{e}_\lambda\right\} \\ &= \mathbb{E}\left\{e^{-\mu H_v}; \tau_v \le \mathbf{e}_\lambda\right\}\mathbb{E}\left\{e^{-\mu X_{\mathbf{e}_\lambda}}; X_{\mathbf{e}_\lambda} > 0\right\} \\ &= \mathbb{E}\left\{e^{-(\mu H_v + \lambda\tau_v)}\right\}\frac{\lambda\frac{d}{d\lambda}\kappa(\lambda,\mu)}{\kappa(\lambda,\mu)} = \frac{\lambda\frac{d}{d\lambda}\kappa(\lambda,\mu)}{\kappa(\lambda,\mu)}e^{-v\kappa(\lambda,\mu)}. \end{aligned}$$

Here the ˜ sign refers to independent copies of the objects, we have used the strong Markov property, and the penultimate equality comes from differentiating Fristedt's formula. Letting $v \to 0$ in the last result confirms that

$$\frac{d}{d\lambda}\int_0^\infty\int_0^\infty e^{-(\lambda t+\mu x)}\frac{\mathbb{P}\{X_t\in dx\}dt}{t}=\frac{-\frac{d}{d\lambda}\kappa(\lambda,\mu)}{\kappa(\lambda,\mu)},$$

so (5.2.4) follows, and hence the result. ■

The discrete version of this identity has been applied in a study of the bivariate renewal function of the ladder process in connection with the Martin boundary of the process killed on leaving the positive half-line ([5]). For Lévy processes, amongst other things Alili and Chaumont deduce in [2] the following identity, which relates the bivariate renewal measure for the increasing ladder processes, $U(dt,dx)$, to the entrance law $\underline{n}$ of the excursions away from 0 of the reflected process $R^*=X-I$:

$$\underline{n}(\varepsilon_t\in dx)dt=ct^{-1}\mathbb{E}(\sigma_x;X_t\in dx)dt=c'U(dt,dx),\ \ x,t>0. \tag{5.2.5}$$

Observe that the equality between the first and last terms is a kind of analogue of an important duality relation for random walks:

$$\begin{aligned}&P(S_r>0,1\le r\le n,S_n\in dx)\\&\qquad=P(n\text{ is an increasing ladder epoch, }S_n\in dx),\end{aligned}$$

which is a “disintegrated” version of (4.2.3) in Chapter 4. Note also that in the case of a spectrally negative Lévy process (see Chapter 9) the middle term in (5.2.5) reduces to $ct^{-1}x\mathbb{P}(X_t\in dx)dt$.

5.3 Les Équations Amicales of Vigon

In his thesis ([100]; see also [99]) Vincent Vigon established a set of equations which essentially invert the Wiener–Hopf factorisation of the exponent

$$\kappa(0,-i\theta)\kappa^*(0,i\theta)=\Psi(\theta),\ \ \theta\in\mathbb{R}. \tag{5.3.1}$$

(We will assume a choice of normalisation of local time that makes the constant which appears on the right-hand side of (5.3.1) in Chapter 4 equal to 1.) Implicit in this equation are relationships between the characteristics of X, H_+, and H_-, which we will denote by $\{\gamma,\sigma^2,\Pi\}$, $\{\delta_+,k_+,\mu_+\}$ and $\{\delta_-,k_-,\mu_-\}$. (n.b. we prefer the notation H_+, and H_-, etc to our standard H and H^* in this section.) We will also write $\phi_\pm$ for the Laplace exponents of $H_\pm$, so that

$$\phi_\pm(\lambda)=k_\pm+\delta_\pm\lambda+\int_0^\infty\left(1-e^{-\lambda x}\right)\mu_\pm(dx),\mathrm{Re}(\lambda)\ge 0,$$

and (5.3.1) is

$$\phi_+(-i\theta)\phi_-(i\theta)=\Psi(\theta),\ \ \theta\in\mathbb{R}. \tag{5.3.2}$$

We will also write Π_+ and Π_- for the restrictions of $\Pi(dx)$ and $\Pi(-dx)$ to $(0,\infty)$, and for any measure Γ on $(0,\infty)$ define tail and integrated tail functions, when they exist, by

$$\overline{\Gamma}(x) = \Gamma\{(x,\infty)\}, \overline{\overline{\Gamma}}(x) = \int_x^\infty \overline{\Gamma}(y)dy, \ x > 0.$$

Theorem 16. *(Vigon) (i) For any Lévy process the following holds:*

$$\overline{\Pi}(x) = \int_0^\infty \mu_+(x+du)\overline{\mu}_-(u) + \delta_- n_+(x) + k_-\overline{\mu}_+(x), \ x > 0, \tag{5.3.3}$$

where, if $\delta_- > 0$, n_+ *denotes a cadlag version of the density of* μ_+*. (It is part of the Theorem that this exists.) Also*

$$\overline{\mu}_+(x) = \int_0^\infty U_-(dy)\overline{\Pi}_+(y+x), \ x > 0. \tag{5.3.4}$$

where U_- *is the renewal measure corresponding to* H_-*.*

(ii) For any Lévy process with $\mathbb{E}|X_1| < \infty$ *the following holds:*

$$\overline{\overline{\Pi}}_+(x) = \int_0^\infty \overline{\mu}_+(x+u)\overline{\mu}_-(u)du + \delta_-\overline{\mu}_+(x) + k_-\overline{\overline{\mu}}_+(x), \ x > 0. \tag{5.3.5}$$

Several comments are in order. Firstly these equations were named by Vigon as the équation amicale, équation amicale inversée, and équation amicale intégrée, respectively. (5.3.4) is equivalent to its differentiated version, and when $\mathbb{E}|X_1| < \infty$ an integrated version holds, but a differentiated version of (5.3.3) only holds in special cases. It should be noted that if we express these equations in terms of H_+ and $-H_-$ as Vigon does, each of the integrals appearing above is in fact a convolution. A version of (5.3.5) can be found in [85], but otherwise these equations don't seem to have appeared in print prior to [99].

From Vigon's standpoint (5.3.3) is just the Fourier inversion of (5.3.2), and (5.3.4) is just the Fourier inversion of the equation

$$\phi_+(-i\theta) = \Psi(\theta) \cdot \frac{1}{\phi_-(i\theta)}, \ \theta \in \mathbb{R}.$$

To make sense of this one needs to use the theory of generalised distributions, but here I give a less technical approach.

Proof. First we aim to establish (5.3.5), and we start by computing the (ordinary) Fourier transform of $f(x) := \overline{\overline{\Pi}}_+(x)\mathbf{1}_{(x>0)} + \overline{\overline{\Pi}}_-(-x)\mathbf{1}_{(x<0)}$. Two integrations by parts give

$$\int_0^\infty \overline{\overline{\Pi}}_+(x)e^{i\theta x}dx = \frac{1}{(i\theta)^2}\int_0^\infty \left(e^{i\theta y} - 1 - i\theta y\right)\Pi(dy),$$

and a similar calculation confirms that

$$\int_{-\infty}^0 \overline{\overline{\Pi}}_-(-x)e^{i\theta x}dx = \frac{1}{(i\theta)^2}\int_{-\infty}^0 \left(e^{i\theta y} - 1 - i\theta y\right)\Pi(dy).$$

Hence the exponent of X satisfies

$$\begin{aligned}\Psi(\theta) &= -i\gamma\theta + \frac{1}{2}\sigma^2\theta^2 + \int_{-\infty}^{\infty} \left(1 - e^{i\theta y} + i\theta y \mathbf{1}_{\{|y|<1\}}\right) \Pi(dy) \\ &= -im\theta + \frac{1}{2}\sigma^2\theta^2 + \int_{-\infty}^{\infty} \left(1 - e^{i\theta y} + i\theta y\right) \Pi(dy) \\ &= -im\theta + \frac{1}{2}\sigma^2\theta^2 + \theta^2 \hat{f}(\theta), \end{aligned} \tag{5.3.6}$$

where $\hat{f}$ is the Fourier transform of f and

$$m = \gamma + \int_{|y|\geq 1} y\Pi(dy) = \mathbb{E}X_1$$

is finite by assumption. Next note that with

$$g_+(\theta) = \int_0^\infty \overline{\mu}_+(x)e^{i\theta x}dx,\ g_-(\theta) = \int_{-\infty}^0 \overline{\mu}_-(-x)e^{i\theta x}dx,$$

we have

$$\phi_+(-i\theta) = k_+ - i\theta\left\{\delta_+ + g_+(\theta)\right\},\ \phi_-(i\theta) = k_- + i\theta\left\{\delta_- + g_-(\theta)\right\}.$$

So, recalling that at most one of $k_\pm$ is non-zero and that $2\delta_+\delta_- = \sigma^2$, it follows from (5.3.2) that

$$\begin{aligned}\Psi(\theta) = \theta^2 &\left\{\sigma^2/2 + g_+(\theta)g_-(\theta) + \delta_+g_-(\theta) + \delta_-g_+(\theta)\right\} \\ &+ i\theta k_+\left\{\delta_- + g_-(\theta)\right\} - i\theta k_-\{\delta_+ + g_+(\theta)\}.\end{aligned} \tag{5.3.7}$$

Substituting this into (5.3.6) we see that for $\theta \neq 0$

$$\begin{aligned}\theta\hat{f}(\theta) - im = \theta&\{g_+(\theta)g_-(\theta) + \delta_+g_-(\theta) + \delta_-g_+(\theta)\} \\ &+ ik_+\{\delta_- + g_-(\theta)\} - ik_-\{\delta_+ + g_+(\theta)\}.\end{aligned} \tag{5.3.8}$$

Further, if $m = \mathbb{E}X_1 = 0$ then X oscillates and $k_+ = k_- = 0$, whence

$$\hat{f}(\theta) = g_+(\theta)g_-(\theta) + \delta_+g_-(\theta) + \delta_-g_+(\theta) \text{ for } \theta \neq 0.$$

Next, assume that $m > 0$, so that X drifts to $+\infty$, $k_+ = 0$, and $k_- > 0$; as we have seen (Chapter 4, Corollary 4) the Wiener–Hopf factorisation gives $m = k_-m_+$, where

$$m_+ := EH_+(1) = \delta_+ + \overline{\overline{\mu}}_+(0+) \in (0, \infty).$$

Thus we then have

$$\overline{\overline{\mu}}_+(0+) - g_+(\theta) = \int_0^\infty \overline{\mu}_+(x)\{1 - e^{i\theta x}\}dx = -i\theta\int_0^\infty \overline{\overline{\mu}}_+(x)e^{i\theta x}dx,$$

so that again the constants in (5.3.8) cancel to give

$$\hat{f}(\theta) = g_+(\theta)g_-(\theta) + \delta_+ g_-(\theta) + \delta_- g_+(\theta) + k_- \int_0^\infty \overline{\overline{\mu}}_+(x)e^{i\theta x}dx. \qquad (5.3.9)$$

Finally a similar argument applies when $m < 0$, and we conclude that in all cases

$$\begin{aligned}\hat{f}(\theta) &= g_+(\theta)g_-(\theta) + \delta_+ g_-(\theta) + \delta_- g_+(\theta)\\ &\quad + k_- \int_0^\infty \overline{\overline{\mu}}_+(x)e^{i\theta x}dx + k_+ \int_{-\infty}^0 \overline{\overline{\mu}}_-(-x)e^{i\theta x}dx. \qquad (5.3.10)\end{aligned}$$

We now observe that $g_+(\theta)g_-(\theta) = \int_{-\infty}^\infty e^{i\theta x} g(x)dx$, where

$$\begin{aligned} g(x) &= \int_{-\infty}^\infty \overline{\mu}_+(x-y)\mathbf{1}_{\{y<x\}}\overline{\mu}_-(-y)dy\\ &= \mathbf{1}_{\{x>0\}}\int_{-\infty}^0 \overline{\mu}_+(x-y)\overline{\mu}_-(-y)dy + \mathbf{1}_{\{x<0\}}\int_{-\infty}^x \overline{\mu}_+(x-y)\overline{\mu}_-(-y)dy\\ &= \mathbf{1}_{\{x>0\}}\int_0^\infty \overline{\mu}_+(x+y)\overline{\mu}_-(y)dy + \mathbf{1}_{\{x<0\}}\int_0^\infty \overline{\mu}_+(y)\overline{\mu}_-(y-x)dy.\end{aligned}$$

Putting this into (5.3.10), and using the uniqueness of Fourier transforms we see that (5.3.5) and its analogue for the negative half-line must hold.

Notice that the left-hand side and the final term on the right-hand side in (5.3.5) are differentiable. Assume, for the moment, the validity of (5.3.4) and assume $\delta_- > 0$; then, according to Theorem 11 of Chapter 4, U_- admits a density u_- which is bounded and continuous on $(0,\infty)$ and has $u_-(0+) > 0$. So we can write (5.3.4) as

$$\overline{\mu}_+(x) = \int_x^\infty u_-(y-x)\overline{\Pi}_+(y)dy, \; x > 0, \qquad (5.3.11)$$

and I claim this implies that $\overline{\mu}_+$ is differentiable on $(0,\infty)$. To see this take $x > 0$ fixed and write

$$\begin{aligned}&\frac{1}{h}\{\overline{\mu}_+(x) - \overline{\mu}_+(x+h)\}\\ &\quad = \frac{1}{h}\left(\int_x^\infty u_-(y-x)\overline{\Pi}_+(y)dy - \int_{x+h}^\infty u_-(y-x-h)\overline{\Pi}_+(y)dy\right)\\ &\quad = \frac{1}{h}\int_x^{x+h} u_-(y-x)\overline{\Pi}_+(y)dy\\ &\qquad - \frac{1}{h}\int_{x+h}^\infty \left(u_-(y-x) - u_-(y-x-h)\right)\overline{\Pi}_+(y)dy.\end{aligned}$$

Clearly the first term here converges to $u_-(0+)\overline{\Pi}_+(x)$ as $h \downarrow 0$, and the following shows that the second term also converges.

$$\begin{aligned}
&\frac{1}{h}\int_{x+h}^{\infty}(u_-(y-x)-u_-(y-x-h))\,\overline{\Pi}_+(y)dy\\
&=\frac{1}{h}\int_{x+h}^{\infty}\Pi(dz)\int_{x+h}^{z}(u_-(y-x)-u_-(y-x-h))\,dy\\
&=\int_{x+h}^{\infty}\Pi(dz)\frac{(U_-(z-x)-U_-(z-x-h)-U_-(h))}{h}\\
&\to\int_{x}^{\infty}\Pi(dz)\,(u_-(z-x)-u_-(0+))\\
&=\int_{x}^{\infty}\Pi(dz)u_-(z-x)-u_-(0+)\overline{\Pi}_+(x).
\end{aligned}$$

Here we have used dominated convergence and the bound

$$|U_-(z-x)-U_-(z-x-h)-U_-(h)|\le 2ch,$$

where c is an upper bound for u_-. A similar argument applies to the left-hand derivative, and we conclude that the second term on the right in (5.3.5) is differentiable, i.e. μ_+ has a density given by

$$n_+(x)=\int_{x}^{\infty}u_-(z-x)\Pi(dz).$$

So the first term must also be differentiable, and, still assuming that $\mathbb{E}|X_1|<\infty$, we deduce that (5.3.3) holds.

However when $\mathbb{E}|X_1|=\infty$ we can consider a sequence $X^{(n)}$ of Lévy processes which have the same characteristics as X except that $X^{(n)}$ has Lévy measure $\Pi^{(n)}(dx)=\Pi(dx)\mathbf{1}_{\{|x|\le n\}}$. Each of them satisfies (5.3.3), and it follows easily that (5.3.3) holds in general. Moreover since the other terms in this are cadlag, when $\delta_->0$ it follows that n_+ can be taken to be cadlag.

So it remains only to prove (5.3.4). To do this we compare two expressions for the overshoot O_y over $y>0$, both of which we have seen before; see Theorem 2, Chapter 2 and Theorem 11, Chapter 4. They are

$$\mathbb{P}(O_y>x)=\int_0^y\overline{\mu}_+(y-z+x)U_+(dz),$$

and

$$\mathbb{P}(O_y>x)=\int_{-\infty}^{y}\overline{\Pi}(y-z+x)V^{(y)}(dz),$$

where in the second we have

$$V^{(y)}(dz)=\int_0^{\infty}\mathbb{P}\{X_t\in dz,\overline{X_t}\le y\}=\int_{z\vee 0}^{y}U_+(dw)U_-(w-dz).$$

Substituting this in and making a change of variable gives

$$\begin{aligned}\int_0^y \overline{\mu}(y-z+x)U_+(dz) &= \int_{-\infty}^y \int_{z\vee 0}^y U_+(dw)U_-(w-dz)\overline{\Pi}(y-z+x)\\ &= \int_0^y \int_{-\infty}^w U_+(dw)U_-(w-dz)\overline{\Pi}(y-z+x)\\ &= \int_{w=0}^y \int_{u=0}^\infty U_+(dw)U_-(du)\overline{\Pi}(y+x+u-w).\end{aligned}$$

For fixed x, the left-hand side here is the convolution of $\overline{\mu}(x+\cdot)$ with U_+ and the right-hand side is the convolution of $h(x+\cdot)$ with U_+, where

$$h(v)=\int_0^\infty U_-(du)\overline{\Pi}(u+v).$$

Using Laplace transforms, we deduce immediately that $\overline{\mu}(x+v)\equiv h(v)$, and this is (5.3.4). ■

These results, particularly (5.3.4), have already found several applications, some of which I will discuss later. Here I will show how (5.3.5) leads to a nice proof of a famous result due to Rogozin (see [88]); this argument is also taken from [100].

Theorem 17. *If X has infinite variation, then*

$$-\infty=\liminf_{t\downarrow 0}\frac{X_t}{t}<\limsup_{t\downarrow 0}\frac{X_t}{t}=+\infty \text{ a.s.} \tag{5.3.12}$$

Proof. We will first establish the weaker claim that any infinite-variation process visits both half-lines immediately. We will also assume without loss of generality that Π is supported by $[-1,1]$, because the compound Poisson process component doesn't affect the behaviour of X immediately after time zero. The argument proceeds by contradiction; so assume X doesn't visit $(0,\infty)$ immediately. This tell us that H_+ is a compound Poisson process, so $\sigma=0$ and $\delta_+=0$. Since both μ_+ and μ_- are supported by $[0,1]$, (5.3.5) for X and $-X$ take the forms

$$\overline{\overline{\Pi}}_+(x)=\int_0^1 \overline{\mu}_+(x+u)\overline{\mu}_-(u)du+\delta_-\overline{\mu}_+(x)+k_-\overline{\overline{\mu}}_+(x),\ x>0,$$

and

$$\overline{\overline{\Pi}}_-(x)=\int_0^1 \overline{\mu}_-(x+u)\overline{\mu}_+(u)du+k_+\overline{\overline{\mu}}_-(x),\ x>0.$$

Since $\overline{\overline{\mu}}_\pm(0+)$ are automatically finite and $\overline{\mu}_+(0+)$ is finite because H_+ is a compound Poisson process, we see immediately that

$$\int_0^1 |x|\Pi(dx)=\overline{\overline{\Pi}}_+(0+)+\overline{\overline{\Pi}}_-(0+)<\infty.$$

This, together with $\sigma = 0$, means that X has bounded variation, and this contradiction establishes the claim. And then (5.3.12) is immediate, because it is equivalent to the fact that for any a, $X_t + at$ visits both half-lines immediately, and of course $X_t + at$ is also an infinite variation Lévy process. ■

5.4 A First Passage Quintuple Identity

We revisit the argument used in Chapter 4 to establish Bertoin's identity for the process killed at time T_x, which played a rôle in our proof of (5.3.4). The corresponding result for random walks is easily established, but again the proof for Lévy process is more complicated.

Recall the notation

$$G_t = \sup\{s \le t : X_s = S_s\},$$

put $\gamma_x = G(T_x-)$ for the time at which the last maximum prior to first passage over x occurs, and denote the overshoot and undershoot of X and undershoot of H_+ by

$$O_x = X(T_x) - x,\ D_x = x - X(T_x-),\ \text{and } D_x^{(H)} = x - S(T_x-).$$

Theorem 18. *Suppose that X is not a compound Poisson process. Then for a suitable choice of normalising constant of the local time at the maximum, for each $x > 0$ we have on $u > 0$, $v \ge y$, $y \in [0, x]$, $s, t > 0$,*

$$\begin{aligned}\mathbb{P}(\gamma_x \in ds, T_x - \gamma_x \in dt, O_x \in du, D_x \in dv, D_x^{(H)} \in dy)\\ = U_+(ds, x - dy)U_-(dt, dv - y)\Pi(du + v),\end{aligned}$$

where $U_\pm$ denote the renewal measures of the bivariate ladder processes.

Proof. (A slightly different proof is given in Doney and Kyprianou, [39].) If we can show the following identity of measures on $(0,\infty)^3$:

$$\begin{aligned}\int_0^\infty qe^{-qt}\mathbb{P}(G_{t-} \in ds,\ S_{t-} \in dw,\ X_{t-} \in w - dz)dt \qquad (5.4.1)\\ = \int_0^\infty qe^{-qt}U_+(ds, dw)U_-(dt - s, dz),\end{aligned}$$

then the result will follow by applying the compensation formula and the uniqueness of Laplace transforms. We establish (5.4.1) by our now standard method: we show their triple Laplace transforms agree. Starting with the left-hand side, we see that it is the same as

$$\begin{aligned}&\mathbb{P}(G_{e_q} \in ds,\ S_{e_q} \in dw,\ X_{e_q} \in w - dz)\\ &\quad = \mathbb{P}(G_{e_q} \in ds,\ S_{e_q} \in dw)\mathbb{P}((S - X)_{e_q} \in dz),\end{aligned}$$

and its triple Lapace transform is

$$\begin{aligned}\mathbb{E}(e^{-\alpha G_{e_q}-\beta S_{e_q}})\mathbb{E}(e^{\gamma I_{e_q}}) &= \frac{\kappa(q,0)}{\kappa(q+\alpha,\beta)}\cdot\frac{\kappa^*(q,0)}{\kappa^*(q,\gamma)}\\ &= \frac{q}{\kappa(q+\alpha,\beta)\kappa^*(q,\gamma)}.\end{aligned}$$

On the other hand,

$$\begin{aligned}&\int_{s,w,z\geq 0}\int_{t\geq s} qe^{-(qt+\alpha s+\beta w+\gamma z)}U_+(ds,dw)U_-(dt-s,dz)\\ &= \int_{s,w,z\geq 0}\int_{u\geq 0} qe^{-(qu+(\alpha+q)s+\beta w+\gamma z)}U_+(ds,dw)U_-(du,dz)\\ &= q\int_{s,w\geq 0} e^{-((\alpha+q)s+\beta w)}U_+(ds,dw)\int_{u,z\geq 0} e^{-(qu+\gamma z)}U_-(du,dz)\\ &= \frac{q}{\kappa(q+\alpha,\beta)\kappa^*(q,\gamma)},\end{aligned}$$

and (5.4.1) follows. ■

One interesting consequence of this is the following obvious extension of (5.3.4); here $\boldsymbol{\mu}_+$ is the bivariate Lévy measure of $\{\tau_+, H_+\}$.

Corollary 6. *For all $t, h > 0$ we have*

$$\boldsymbol{\mu}_+(dt,dh) = \int_{[0,\infty)} U_-(dt,d\theta)\Pi(dh+\theta).$$

A second is a new explicit result for stable processes, whose proof relies on the well-known fact that in this case the subordinators H_+, H_- are stable with parameters $\alpha\rho, \alpha(1-\rho)$, respectively.

Corollary 7. *Let X be a stable process of index $\alpha \in (0,2)$ and positivity parameter $\rho \in (0,1)$. Then*

$$\begin{aligned}&\mathbb{P}(O_x \in du, D_x \in dv, D_x^{(H)} \in dy)\\ &= \frac{\Gamma(\alpha+1)\sin\alpha\rho\pi}{\pi\Gamma(\alpha\rho))\,\Gamma(\alpha(1-\rho))}\cdot\frac{(x-y)^{\alpha\rho-1}(v-y)^{\alpha(1-\rho)-1}du\,dv\,dy}{(v+u)^{1+\alpha}}.\end{aligned}$$

A further application, indeed the main motivation in [39], is a study of the asymptotic overshoot over a high level, conditional upon this level being crossed, for a class of processes which drift to $-\infty$ and whose Lévy measures have exponentially small righthand tails. It was already known from [66] that there is a limiting distribution for this overshoot, which has two components. Using Theorem 18 we were able to show that these components are the consequence of two different types of asymptotic overshoot: namely first passage occurring as a result of

- an arbitrarily large jump from a finite position after a finite time, or
- a finite jump from a finite distance relative to the barrier after an arbitrarily large time.

6

Creeping and Related Questions

6.1 Introduction

We have seen that a subordinator creeps over positive levels if and only if it has non-zero drift. Since the overshoot over a positive level of a Lévy process X coincides with the overshoot of its increasing ladder height subordinator, it is clear that X creeps over positive levels if and only if the drift δ_+ of H_+ is positive. This immediately raises the question as to how one can tell, **from the characteristics of** X, when this happens. This question was first addressed in Millar [77], where the concept of creep was introduced, although actually Millar called it continuous upward passage. Some partial answers were given in Rogers [85], where the name "creeping" was first introduced, but the complete solution is due to Vigon [99], [100]. Another reason why the condition $\delta_+ > 0$ is important is that we will see in Chapter 10 that it is also a necessary and sufficient condition for

$$\frac{O_r^{(X)}}{r} = \frac{O_r^{(H_+)}}{r} \stackrel{a.s.}{\to} 0$$

as $r \downarrow 0$. (Here $O_r^{(X)} = X_{T(r,\infty)} - r$, and similarly for H_+.) Of course a necessary and sufficient condition for this to hold as $r \to \infty$ is that

$$m_+ = \mathbb{E}H_+(1) = \delta_+ + \int_0^\infty \overline{\mu}_+(x)dx < \infty,$$

and similarly one can ask how we can recognise when this happens from the characteristics of X. For random walks, this 'mean ladder height problem' has been around for a long time; after contributions by Lai [71], Doney [29], and Chow and Lai [27], it was finally solved in Chow [26]. This last paper passed almost unnoticed, which is a pity because on the basis of Chow's result it is easy to see what the result for Lévy process has to be at ∞, and not difficult to guess also what the result should be at 0. In [40] we used Chow's result to give the necessary and sufficient condition for $m_+ < \infty$, but somehow we managed to make a wrong conjecture for $\delta_+ > 0$!

Here I will give a proof of both results, using a method that leans heavily on results from Vigon [100], but is somewhat different from the proof therein. We will also see that the same techniques enable us to give a different proof of an important result in Bertoin [14], which solves the problem of regularity of the half-line.

6.2 Notation and Preliminary Results

As usual X will be a Lévy process with Lévy measure Π, and having canonical decomposition

$$X_t = \gamma t + \sigma B_t + Y_t^{(1)} + Y_t^{(2)}. \tag{6.2.1}$$

We write $\mu_\pm, \delta_\pm$, and $k_\pm$ for the Lévy measures, drifts and killing rates for $H_\pm$, the ladder height processes of X and $-X$. We will also need $U_\pm$, the potential measures of $H_\pm$. The basis for our whole approach is Vigon's "équation amicale inversée", which we recall from Chapter 5 is

$$\overline{\mu}_+(x) = \int_0^\infty U_-(dy)\overline{\Pi}_+(x+y),\ x > 0. \tag{6.2.2}$$

The second result we need is a slight extension of one we've seen before, in Chapter 2; here and throughout, we write $a(x) \approx b(x)$ to signify that $\exists$ absolute constants $0 < C_1 < C_2 < \infty$ with $C_1 \le a(x)/b(x) \le C_2$ for all $x \in (0, \infty)$ and write C for a generic positive absolute constant.

Lemma 4. *If U is the renewal function of any subordinator having killing rate k, drift δ, and Lévy measure μ, and*

$$A(x) = \delta + \int_0^x \overline{\mu}(y)dy,$$

then

$$U(x) \approx \frac{x}{A(x) + kx}. \tag{6.2.3}$$

This result first appeared in Erickson [45] in the context of renewal processes, and we used it in Chapter 4; see (4.5.18) therein. For subordinators it appears as Proposition 1, p. 33, of [12]. In both these references k is taken to be zero, but the extension to the case $k \neq 0$, which is given in [26] for renewal processes and [100] for subordinators, is straightforward.

Lemma 5. *Writing $A_+(x) = \delta_+ + \int_0^x \overline{\mu}_+(y)dy$ and Π^* for the Lévy measure of $-X$, we have*

$$A_+(x) \approx \delta_+ + \int_0^\infty \frac{t(t_\wedge x)\Pi(dt)}{\delta_- + k_- t + \int_0^\infty \frac{z(z_\wedge t)\Pi^*(dz)}{k_+ z + A_+(z)}}. \tag{6.2.4}$$

Proof. We can rewrite (6.2.2) as

$$\begin{aligned}\overline{\mu}_+(x) &= \int_0^\infty U_-(dy)\int_{z>x+y}\Pi(dz) = \int_x^\infty \Pi(dz)\int_{y<z-x}U_-(dy)\\ &= \int_x^\infty U_-(z-x)\Pi(dz), \qquad (6.2.5)\end{aligned}$$

and putting this into the definition of A_+ we get

$$\begin{aligned}A_+(x) &= \delta_+ + \int_0^x du\int_u^\infty U_-(z-u)\Pi(dz)\\ &= \delta_+ + \int_0^\infty \Pi(dz)\int_0^{z_\wedge x} U_-(z-u)du.\end{aligned}$$

Using (6.2.3) we have the bounds

$$\int_0^{z_\wedge x} U_-(z-u)du \le (z_\wedge x)U_-(z) \le C\frac{z(z_\wedge x)}{k_- z + A_-(z)}$$

and

$$\begin{aligned}\int_0^{z_\wedge x} U_-(z-u)du &= \int_{z-z_\wedge x}^z U_-(v)dv \ge C\int_{z-z_\wedge x}^z \frac{v}{k_- v + A_-(v)}dv\\ &\ge \frac{C}{k_- z + A_-(z)}\int_{z-z_\wedge x}^z vdv \ge C\frac{z(z_\wedge x)}{k_- z + A_-(z)}.\end{aligned}$$

These yield

$$A_+(x) \approx \delta_+ + \int_0^\infty \frac{t(t_\wedge x)\Pi(dt)}{k_- t + A_-(t)}. \qquad (6.2.6)$$

Now we feed back into this the same result for A_-, and we get (6.2.4). (This device is due to Chow [26].) ■

6.3 The Mean Ladder Height Problem

We are only interested in the case when H_+ has infinite lifetime, so in this section we will have $k_+ = 0$. Note first that $A_\pm$ are truncated means, in the sense that

$$\lim_{x\to\infty} A_\pm(x) = m_\pm \le \infty.$$

Also $A_\pm(x)$ are $o(x)$ as $x \to \infty$, so if $k_- > 0$, which happens if and only if X drifts to $+\infty$, letting $x \to \infty$ in (6.2.6) we see that $m_+ < \infty$ if and only if $\int_1^\infty t\Pi(dt) < \infty$, i.e. $\mathbb{E}X_1 < \infty$. Thus we can take $k_- = 0$, so that X oscillates. The same argument shows that $\mathbb{E}X_1 = \infty$ implies $m_+ = \infty$, so we can take $\mathbb{E}|X_1| < \infty$ and $\mathbb{E}X_1 = 0$. In this case it is convenient to introduce

$$G_+(x) = \int_x^\infty y\Pi(dy), \ G_-(x) = \int_x^\infty y\Pi^*(dy), \tag{6.3.1}$$

and note that

$$\int_0^\infty z.(z_\wedge t)\Pi^*(dz) = \int_0^\infty z\Pi^*(dz)\int_0^{z_\wedge t} dy = \int_0^t G_-(y)dy. \tag{6.3.2}$$

Theorem 19. *Let X be any Lévy process having $\mathbb{E}|X_1| < \infty$ and $\mathbb{E}X_1 = 0$: then m_+ is finite if and only if*

$$I = \int_1^\infty \frac{t^2\Pi(dt)}{\int_0^\infty z.(z_\wedge t)\Pi^*(dz)} = \int_1^\infty \frac{-tdG_+(t)}{\int_0^t G_-(z)dz} < \infty. \tag{6.3.3}$$

Proof. First recall that in Chapter 4, Corollary 4 we showed that in these circumstances we have $2m_+m_- = \mathbb{E}X^2 \leq \infty$, and note that $\mathbb{E}X^2 < \infty \Longrightarrow I < \infty$. So from now on assume $\mathbb{E}X^2 = \infty$, in which case at most one of m_+ and m_- is finite. Suppose next that $m_+ = A_+(\infty) < \infty$; then $m_- = \infty$, and so for any $x_0 \in (0,\infty)$

$$\begin{aligned} A_-(x) &\sim \int_0^x \overline{\mu}_-(y)dy \sim \int_0^{x+x_0} \overline{\mu}_-(y)dy \\ &\sim \int_0^x \overline{\mu}_-(y+x_0)dy \text{ as } x \to \infty. \end{aligned}$$

Now choose x_0 such $c := \overline{\mu}_+(x_0) > 0$ and use Vigon's équation amicale intégrée (5.3.5) for $-X$ to get

$$\begin{aligned} \overline{\overline{\Pi^*}}(x) &= \int_0^\infty \overline{\mu}_-(y+x)\overline{\mu}_+(y)dy + \delta_+\overline{\mu}_-(x) \\ &\geq c\int_0^{x_0} \overline{\mu}_-(y+x)dy \geq cx_0\overline{\mu}_-(x+x_0), \end{aligned}$$

so that

$$A_-(x) \sim \int_0^x \overline{\mu}_-(y+x_0)dy \leq (cx_0)^{-1}\int_0^x \overline{\overline{\Pi^*}}(y)dy.$$

Hence, letting $x \to \infty$ in (6.2.6) gives

$$\int_0^\infty \frac{t^2\Pi(dt)}{\int_0^t \overline{\overline{\Pi^*}}(y)dy} < \infty,$$

and since $\overline{\overline{\Pi^*}}(y) \leq G_-(y)$ this implies $I < \infty$. To argue the other way we assume $I < \infty$ and $m_+ = \infty$, and establish a contradiction by showing that $I_b \nrightarrow 0$ as $b \to \infty$, where

$$I_b = \int_b^\infty \frac{t^2\Pi(dt)}{\int_0^\infty z.(z_\wedge t)\Pi^*(dz)}. \tag{6.3.4}$$

Let $X^{(\varepsilon)}$ denote a Lévy process with the same characteristics as X except that

$$\Pi^{(\varepsilon)}(dx) = \Pi(dx) + \varepsilon\delta_1(dx),$$

where $\varepsilon > 0$ and $\delta_1(dx)$ denotes a unit mass at 1. Clearly $X^{(\varepsilon)}$ drifts to $+\infty$, so $k_-^{(\varepsilon)} > 0 = k_+^{(\varepsilon)}$, and $m_+^{(\varepsilon)} < \infty$, because $\mathbb{E}|X_1^{(\varepsilon)}| < \infty$. Also $\delta_\pm^{(\varepsilon)} = \delta_\pm$, and $k_-^{(\varepsilon)} \to k_- = 0$, $m_+^{(\varepsilon)} \to m_+ = \infty$ as $\varepsilon \downarrow 0$. Now take $b > 1$ fixed, apply (6.2.4) to $X^{(\varepsilon)}$ and let $x \to \infty$ to get

$$m_+^{(\varepsilon)} = A_+^{(\varepsilon)}(\infty) \le C\{\delta_+ + I_\varepsilon^{(1)} + I_\varepsilon^{(2)}\}$$

with

$$\begin{aligned}\delta_+ + I_\varepsilon^{(1)} &= \delta_+ + \int_0^b \frac{t^2\Pi^{(\varepsilon)}(dt)}{\delta_- + k_-^{(\varepsilon)}t + \int_0^\infty \frac{z(z_\wedge t)\Pi^*(dz)}{A_+^{(\varepsilon)}(z)}} \\ &\le \delta_+ + \int_0^\infty \frac{t(t_\wedge b)\Pi^{(\varepsilon)}(dt)}{\delta_- + k_-^{(\varepsilon)}t + \int_0^\infty \frac{z(z_\wedge t)\Pi^*(dz)}{A_+^{(\varepsilon)}(z)}} \le CA_+^{(\varepsilon)}(b),\end{aligned}$$

where we have used (6.2.4) again. Also, using $A_+^{(\varepsilon)}(z) \le A_+^{(\varepsilon)}(\infty) = m_+^{(\varepsilon)}$ we have

$$I_\varepsilon^{(2)} \le m_+^{(\varepsilon)} \int_b^\infty \frac{t^2\Pi(dt)}{\int_0^\infty z.(z_\wedge t)\Pi^*(dz)} = m_+^{(\varepsilon)} I_b,$$

so we have shown that

$$m_+^{(\varepsilon)} \le C\{A_+^{(\varepsilon)}(b) + m_+^{(\varepsilon)} I_b\}.$$

Since $m_+^{(\varepsilon)} \to \infty$ and $A_+^{(\varepsilon)}(b) \to A_+(b) < \infty$ as $\varepsilon \downarrow 0$, we conclude that $I_b \ge 1/C$ for all $b > 1$, and the result follows. ■

This proof is actually simpler than that for the random-walk case in [26]: moreover by considering the special case of a compound Poisson process, Theorem 19 implies Chow's result.

There is an obvious, but puzzling, connection between the integral test in Theorem 19 and the Erickson result, Theorem 15 in Chapter 4. Specifically, if X is a Lévy process satisfying

$$\mathbb{E}|X_1| < \infty,\ \mathbb{E}X_1 = 0,\ \int_0^\infty x^2\Pi^*(dx) = \int_0^\infty x^2\Pi(dx) = \infty, \qquad (6.3.5)$$

we can define another Lévy process $\tilde{X}$ with $\tilde{\Pi}(dx) = |x|\Pi(dx)$ which has $\mathbb{E}\tilde{X}_1^+ = \mathbb{E}\tilde{X}_1^- = \infty$, and this process satisfies $t^{-1}\tilde{X}_t \to \infty$ a.s. as $t \to \infty$ if and only if $m_+ = \infty$.

6.4 Creeping

Let us first dispose of some easy cases. As we have seen (Corollary 4, Chapter 4) $\sigma^2 = 2\delta_+\delta_-$, so we will take $\sigma^2 = 0$; then at least one of these drifts has to be 0. If X has bounded variation, it has true drift

$$\tilde{\gamma} = \gamma - \int_{\{|x|\le 1\}} x\Pi(dx),$$

(i.e. $t^{-1}X_t \to \tilde{\gamma}$ as $t \downarrow 0$), and this is similar to the subordinator case: $\delta_+ > 0$ if and only if $\tilde{\gamma} > 0$. Also, in the decomposition (6.2.1), the compound Poisson term $Y^{(2)}$ has no effect on whether X creeps, since it is zero until the time at which the first 'large jump' occurs. So we can assume that Π is concentrated on $[-1,1]$ with $\int_{-1}^{1} |x|\Pi(dx) = \infty$, and further, by altering the mass at ± 1, that $\mathbb{E}X_1 = 0$. Thus X oscillates, $k_+ = k_- = 0$, and (6.3.1) reduces to

$$G_+(x) = \int_x^1 y\Pi(dy),\ G_-(x) = \int_x^1 y\Pi^*(dy).$$

Theorem 20. *(Vigon) Assume that X has infinite variation; then $\delta_+ > 0$ if and only if*

$$J = \int_0^1 \frac{t^2\Pi(dt)}{\int_0^1 z.(z_\wedge t)\Pi^*(dz)} = \int_0^1 \frac{-tdG_+(t)}{\int_0^t G_-(z)dz} < \infty.$$

Proof. As remarked above we can assume that the support of Π is contained in $[-1,1]$ and $\mathbb{E}X_1 = 0$. Note first that $\delta_+ > 0 \Longrightarrow \delta_- = 0$ and $A_+(z) \ge \delta_+$; putting this into (6.2.4) with $x = 1$ yields

$$A_+(1) \ge c\int_0^1 \frac{t^2\Pi(dt)}{\frac{1}{\delta_+}\int_0^1 z.(z_\wedge t)\Pi^*(dz)} = c\delta_+ J,$$

so that $\delta_+ > 0 \Longrightarrow J < \infty$. To argue the other way, we consider first the case that $G_+(0+) < \infty$. Here we claim that we always have $J < \infty$ and $\delta_+ > 0$. The first follows because $G_-(0+) = \infty$ (otherwise we would be in the bounded variation case), so that

$$\frac{t}{\int_0^t G_-(z)dz} = o(1) \text{ as } t \downarrow 0.$$

For the second observe that, in the notation of Chapter 5, we have $\overline{\overline{\overline{\Pi}}}_+(0+) < \infty = \overline{\overline{\overline{\Pi}}}_-(0+)$. We can therefore let $x \downarrow 0$ in (5.3.5) to see that

$$\int_0^1 \overline{\mu}_+(u)\overline{\mu}_-(u)du = \lim_{x\downarrow 0}\int_0^1 \overline{\mu}_+(x+u)\overline{\mu}_-(u)du < \infty.$$

But if we had $\delta_+ = 0$ it would follow from (5.3.5) for $-X$ and monotone convergence that

$$\overline{\overline{\Pi}}_-(0+) = \lim_{x \downarrow 0} \int_0^1 \overline{\mu}_-(x+u)\overline{\mu}_+(u)du < \infty.$$

This contradiction shows that $\delta_+ > 0$. (This argument is taken from Rogers [85], although the original result is in Millar [77].) Now assume that $G_+(0+) = \infty$ and $\delta_+ = 0$: then we have, for $z \leq 1$,

$$A_+(z) \leq A_+(1) = \int_0^1 \overline{\mu}_+(y)dy < \infty.$$

Putting this into (6.2.4) again with $x = 1$ yields

$$A_+(1) \leq C \int_0^1 \frac{t^2 \Pi(dt)}{\frac{1}{A_+(1)} \int_0^1 z.(z_\wedge t)\Pi^*(dz)} = CA_+(1)J,$$

so that

$$J \geq c = \frac{1}{C}. \tag{6.4.1}$$

It is important to note that (6.4.1) holds for all X satisfying our assumptions, and that c can be taken as an absolute constant. To show that actually $J = \infty$ we consider another Lévy process $\tilde{X}^{(\varepsilon)}$ with the same characteristics as X except that Π is replaced by

$$\tilde{\Pi}^{(\varepsilon)}(dx) = \mathbf{1}_{[-1,\varepsilon]}\Pi(dx) + \varepsilon^{-1}G_+(\varepsilon)\delta_\varepsilon(dx),$$

where δ_ε denotes a unit mass at ε. Note first that

$$\mathbb{E}X_1 = \int_{-1}^{\varepsilon} x\Pi(dx) + G_+(\varepsilon) = \int_{-1}^{1} x\Pi(dx) = 0,$$

so that $\tilde{X}^{(\varepsilon)}$ oscillates, and clearly it has infinite variation. Since Π and $\tilde{\Pi}^{(\varepsilon)}$ agree on $[-1, \varepsilon/2]$, and X does not creep upwards, neither does $\tilde{X}^{(\varepsilon)}$, so if J_ε denotes J evaluated for $\tilde{X}^{(\varepsilon)}$, we have

$$J_\varepsilon = \int_0^\varepsilon \frac{-tdG_+(t)}{\int_0^t G_-(z)dz} + \frac{\varepsilon G_+(\varepsilon)}{\int_0^\varepsilon G_-(z)dz} \geq c. \tag{6.4.2}$$

Suppose now that J is finite; then the first term in (6.4.2) $\to 0$ as $\varepsilon \downarrow 0$. It follows then that $\exists \varepsilon_0 > 0$ such that

$$\int_0^\varepsilon G_-(z)dz \leq \frac{2}{c}\varepsilon G_+(\varepsilon) \text{ for all } \varepsilon \in (0, \varepsilon_0].$$

But then

$$\int_0^{\varepsilon_0} \frac{-tdG_+(t)}{\int_0^t G_-(z)dz} \geq \frac{c}{2} \lim_{\varepsilon \downarrow 0} \int_\varepsilon^{\varepsilon_0} \frac{-tdG_+(t)}{tG_+(t)}$$
$$= \frac{c}{2} \lim_{\varepsilon \downarrow 0} \log \frac{G_+(\varepsilon)}{G_+(\varepsilon_0)} = \infty,$$

because $G_+(0+) = \infty$. This contradiction proves that $J = \infty$. ■

There are several other integrals whose convergence is equivalent to that of J in Theorem 20, and similar remarks apply to I in Theorem 19. To see this, note that

$$\overline{\overline{\Pi^*}}(x) = \int_x^1 \overline{\Pi^*}(z)dz = G_-(x) - x\overline{\Pi^*}(x),$$

so that

$$\int_0^x G_-(t)dt \geq \int_0^x \overline{\overline{\Pi^*}}(z)dz.$$

On the other hand

$$\int_0^x G_-(t)dt = \int_0^x \overline{\overline{\Pi^*}}(z)dz + \int_0^x -zd\overline{\overline{\Pi^*}}(z)$$
$$= 2\int_0^x \overline{\overline{\Pi^*}}(z)dz - x\overline{\overline{\Pi^*}}(x) \leq 2\int_0^x \overline{\overline{\Pi^*}}(z)dz, \qquad (6.4.3)$$

so we can replace $\int_0^x G_-(t)dt$ by $\int_0^x \overline{\overline{\Pi^*}}(z)dz$ in J. By a further integration by parts

$$\tilde{J} := \int_0^1 \frac{x\overline{\Pi}(x)dx}{\int_0^x \overline{\overline{\Pi^*}}(z)dz}$$
$$= \frac{1}{2}\int_0^1 x^2 \frac{\Pi(dx)}{\int_0^x \overline{\overline{\Pi^*}}(z)dz} + \frac{1}{2}\int_0^1 x^2 \frac{\overline{\Pi}(x)\overline{\overline{\Pi^*}}(x)dx}{\left(\int_0^x \overline{\overline{\Pi^*}}(z)dz\right)^2}$$
$$\leq J + \frac{1}{2}\tilde{J},$$

so we see that we can replace J by $\tilde{J}$ in Theorem 20. This is the form given in Vigon [99].

This result has several interesting consequences, all of which are taken from Vigon [100]. First, it implies the following result from Rogers [85]:

Corollary 8. *Suppose X is a Lévy process with infinite variation and no Brownian component satisfying*

$$\liminf_{x \downarrow 0} \frac{\int_x^1 \overline{\Pi}(z)dz}{\int_x^1 \overline{\Pi^*}(z)dz} > 0.$$

Then $\delta_+ = 0$.

Another application is:

Corollary 9. *Suppose X is any Lévy process with infinite variation, and $\hat{X}$ denotes the Lévy process defined by*

$$\hat{X}_t = X_t + \gamma t,$$

where γ is any real constant. Then X creeps upwards if and only if $\hat{X}$ creeps upwards.

This result seems almost obvious, but sample-path arguments do not seem to work. Although this result is from Vigon [100], there an analytic proof is given, and what for us is a corollary of Theorem 20 in his approach is a key lemma in the proof of that Theorem. In a sense the device of considering $\tilde{X}^{(\varepsilon)}$, which is similar to what we did to prove Theorem 19 (which in turn was inspired by Chow [26]), replaces Vigon's proof of this corollary.

Just as we mentioned in connection with Theorem 19, there is a formal similarity between the result in Theorem 20 and another integral test, this time that of Bertoin [14]; see Theorem 22 later in this chapter. Given any Lévy process X which has no Brownian component, we write it as $Y^+ - Y^-$, where $Y^{\pm}$ are independent, spectrally positive Lévy processes, having Lévy measures $\mathbf{1}_{\{x>0\}}\Pi(dx)$ and $\mathbf{1}_{\{x>0\}}\Pi^*(dx)$ respectively. Denote by $H^{\pm}$ the increasing ladder processes of $Y^{\pm}$; (n.b. these are different from $H_{\pm}$, which are the increasing ladder processes of X and $-X$). Then the decreasing ladder processes for $Y^{\pm}$ are pure drifts, possibly killed at an exponential time. Using this fact in (6.2.2), we see that the Lévy measures of $Y^{\pm}$ satisfy

$$\overline{\mu^+}(x) \sim \overline{\overline{\Pi}}(x), \ \overline{\mu^-}(x) \sim \overline{\overline{\Pi^*}}(x) \text{ as } x \downarrow 0.$$

We deduce that

$$\int_0^1 \frac{x\mu^+(dx)}{\int_0^x \overline{\mu^-}(z)dz} < \infty$$

if and only if $\tilde{J} < \infty$. Note that H^+ and H^- both have zero drift, so Bertoin's criterion applies, and we see that X creeps upwards if and only if

$$\lim_{t\downarrow 0} \frac{H_t^+}{H_t^-} = 0.$$

6.5 Limit Points of the Supremum Process

In this section we will write S_t for $\sup_{s\leq t} X_s$, and will be interested in two different behaviours that the paths of S can have: either the (monotone, cadlag) paths have a finite number of jumps in each finite time interval (we will refer to this as Type I behaviour), or the jump times have limit points; we will refer to this as Type II behaviour. Clearly Type I behaviour occurs if and only if the Lévy measure μ_+ of H_+ is a finite measure, so that H_+ is a compound Poisson process with a possible drift δ_+; when this happens it is obvious that $\delta_+ > 0$ occurs if and only if X visits $(0, \infty)$ immediately. If the restriction of Π to $(0, \infty)$ is a finite measure, we will get Type I behaviour, but it is not clear whether this can happen in other cases. The following result, taken from Vigon [100], shows how we can determine which of the two cases occurs.

Theorem 21. *Type I behaviour occurs if and only if one of the following holds:*

1. $\sigma^2 > 0$, *and* $\int_0^1 x\Pi(dx) < \infty$.
2. *X has infinite variation,* $\sigma^2 = 0$, *and*
$$\int_0^1 \frac{x\Pi(dx)}{\int_0^x \overline{\overline{\Pi^*}}(y)dy} < \infty. \tag{6.5.1}$$
3. *X has bounded variation with drift* $\delta > 0$ *and*
$$\int_0^1 \Pi(dx) < \infty.$$
4. *X has bounded variation with drift* $\delta = 0$ *and X does not visit* $(0, \infty)$ *immediately, i.e.*
$$\int_0^1 \frac{x\Pi(dx)}{\int_0^x \overline{\Pi^*}(y)dy} < \infty. \tag{6.5.2}$$
5. *X has bounded variation with drift* $\delta < 0$.

Proof. First note that, letting $x \downarrow 0$ in (6.2.5) and then using (6.2.3), we always have

$$\overline{\mu_+}(0+) < \infty \text{ if and only if } \int_0^\infty \frac{x\Pi(dx)}{\delta_- + k_- x + \int_0^x \overline{\mu}_-(dz)} < \infty. \tag{6.5.3}$$

However, since Type I behaviour is determined by the behaviour of X immediately after time zero, we can alter Π away from 0 without affecting this behaviour, so we can assume that Π is supported on $[-1, 1]$ and $\mathbb{E}X_1 = 0$, and have $k_- = k_+ = 0$. When $\sigma^2 > 0$, $\delta_- > 0$, and the result follows immediately from (6.5.3). If $\sigma^2 = 0$ and X has infinite variation, X visits $(0, \infty)$ immediately, so Type I behaviour implies that $\overline{\mu_+}(0+) < \infty$ and $\delta_+ > 0$, and hence $\delta_- = 0$. Since $U_+(x) \sim x/\delta_+$ as $x \downarrow 0$, an easy consequence of (6.2.5) applied to $-X$ is that

$$\delta_+ \overline{\mu_-}(x) \sim \overline{\overline{\Pi^*}}(x) \text{ as } x \downarrow 0, \tag{6.5.4}$$

so the convergence of the integral in (6.5.1) follows from (6.5.3). On the other hand, from (6.4.3), we clearly have

$$\int_0^1 \frac{x\Pi(dx)}{\int_0^x \overline{\overline{\Pi^*}}(y)dy} \geq \int_0^1 \frac{x^2\Pi(dx)}{\int_0^x \overline{\overline{\Pi^*}}(y)dy} \geq J/3,$$

so if the integral in (6.5.1) converges, X creeps upwards, (6.5.4) again applies, and since $\delta_- = 0$, (6.5.3) shows that $\overline{\mu_+}(0+) < \infty$. In case 3 the assumption that X has bounded variation and $\delta > 0$ implies that $\delta_+ > 0$, (6.5.4) again applies, and since $\delta_- = 0$, the result follows from (6.5.3). Next, if $\delta \leq 0$ and X does not visit $(0, \infty)$ immediately (this is automatic if $\delta < 0$), then H_+ is a compound Poisson process and we have Type I behaviour. On the other hand if $\delta = 0$ and X does visit $(0,\infty)$ immediately, we have $\delta_+ = 0$, and so H_+ is not a compound Poisson process and we do not have Type I behaviour. Finally the integral criterion in (6.5.2) comes from Bertoin [14]; we will prove this in the next section. ■

Corollary 10. *If* $\int_0^1 x\Pi(dx) = \infty$ *then* S *has Type II behaviour.*

Example 1. *If* Y *is a bounded variation Lévy process and* W *is an independent Brownian motion then the supremum and infimum processes of* $X = Y + W$ *both have Type I behaviour. (Somehow the Brownian motion oscillations hide all but a few of the jumps in* Y*.)*

Example 2. *Suppose* $X = Y^+ - Y^-$, *where* $Y^{\pm}$ *are independent, spectrally positive stable processes with parameters* $\alpha^{\pm}$, *respectively. Then we can check that* X *creeps upwards if and only if* $\alpha^+ < \alpha^-$, *but* S *has Type I behaviour if and only if* $1 + \alpha^+ < \alpha^- \in [1, 2)$. *This shows that the converse to Corollary 10 is false.*

6.6 Regularity of the Half-Line

The criterion of Rogozin for regularity of the positive half-line which appeared in Corollary 4, Chapter 4, is not expressed in terms of the characteristics of X. This problem remained open for the case of bounded variation processes till it was solved in Bertoin [14]. His proof is very interesting, but here we show how it can be achieved by the methods of this chapter.

Theorem 22. *(Bertoin) Suppose that* X *has bounded variation: then* 0 *is regular for* $(0, \infty)$ *if and only if* $\delta > 0$, *or*

$$\delta = 0 \text{ and } I = \int_0^1 \frac{x\Pi(dx)}{\int_0^x \overline{\Pi^*}(y)dy} = \infty. \tag{6.6.1}$$

Note that the result is formally a small-time version of Erickson's theorem. The similarity in the results is more obvious if we note that irregularity of $(0, \infty)$ means that X is a.s. negative in a neighbourhood of 0, and drift to $-\infty$ means that X is a.s. negative in a neighbourhood of ∞. Note also that a proof similar to that which follows can be given for Erickson's theorem.

Proof of Theorem 22. The result when $\delta > 0$ is immediate from the strong law at zero, so assume that $\delta = 0$. Since changing Π outside $(-1, 1)$ does not affect the finiteness of I, nor regularity, without loss of generality we can assume that Π is supported by $[-1,1]$, that $\Pi([-1,1]) = \infty$, and $\mathbb{E}X_1 = 0$. In one direction the proof is immediate, because from the équation amicale inversée for $-X$ we see that for any $\eta \in (0,1)$,

$$\begin{aligned}\overline{\mu}_-(x) &= \int_0^{1-x} \overline{\Pi}_-(x+y)U_+(dy) \\ &\leq c_1\overline{\Pi}_-(x) \text{ for all } x \in (0,\eta].\end{aligned}$$

(Here $c_1 = U_+([0,1))$.) We know $\delta_\pm = k_\pm = 0$, so using Lemma 4 we see that

$$\begin{aligned}\int_0^\eta \frac{y\Pi(dy)}{\int_0^y \overline{\Pi}_-(z)dz} &\leq c_2 \int_0^\eta \frac{y\Pi(dy)}{\int_0^y \overline{\mu}_-(z)dz} \\ &\leq c_3 \int_0^\eta U_-(y)\Pi(dy) = c_3 \int_0^\eta U_-(dy)\overline{\Pi}(y).\end{aligned}$$

Now 0 being irregular for $(0,\infty)$ is equivalent to H_+ being a compound Poisson process, i.e. $\overline{\mu}_+(0+) < \infty$. From the équation amicale inversée we see this is equivalent to

$$\int_0^\eta U_-(dy)\overline{\Pi}(y) = \lim_{x\downarrow 0} \int_0^\eta U_-(x+dy)\overline{\Pi}(y) < \infty,$$

so irregularity of the half-line implies $I < \infty$. To argue the other way we suppose $\overline{\mu}_+(0+) = \infty$ and $I < \infty$, and establish a contradiction. Note first that whenever Π is concentrated on $[-1,1]$ and $\mathbb{E}X_1 = 0$ we can use the argument in Lemma 5 to get

$$\begin{aligned}\overline{\mu}_+(x) &= \int_0^{1-x} U_-(z)\Pi(x+dz) \approx \int_0^{1-x} \frac{z\Pi(x+dz)}{A_-(z)} \\ &\approx \int_0^{1-x} \frac{z\Pi(x+dz)}{\int_0^1 \frac{t(t\wedge z)\Pi^*(dt)}{A_+(t)}} = \int_x^1 \frac{(z-x)\Pi(dz)}{\int_0^1 \frac{t(t\wedge(z-x))\Pi^*(dt)}{A_+(t)}}.\end{aligned}$$

We will apply this to $X^{(\varepsilon)} = \{X_t - \varepsilon t + \varepsilon Y_t, t \geq 0\}$, where X is as in the first part of the proof and Y is an independent unit rate Poisson process, so that $\mathbb{E}X_1^{(\varepsilon)} = 0$. Note that $\delta_-^{(\varepsilon)} > 0$, and $\delta_+^{(\varepsilon)} = 0$, so $(0,\infty)$ is irregular for $X^{(\varepsilon)}$, and $\overline{\mu}_+^{(\varepsilon)}(0+) < \infty$. So the above estimate applies to $X^{(\varepsilon)}$ and gives

$$\overline{\mu}_+^{(\varepsilon)}(x) \approx \int_x^1 \frac{(z-x)\Pi(dz)}{\int_0^1 \frac{t(t\wedge(z-x))\Pi^*(dt)}{A_+^{(\varepsilon)}(t)}},$$

and

$$\overline{\mu}_+^{(\varepsilon)}(0+) \approx \int_0^1 \frac{z\Pi(dz)}{\int_0^1 \frac{t(t\wedge z)\Pi^*(dt)}{A_+^{(\varepsilon)}(t)}}.$$

Now take any $0 < b < 1/2$ and note that

$$\int_{2b}^{1} \frac{z\Pi(dz)}{\int_0^1 \frac{t(t\wedge z)\Pi^*(dt)}{A_+^{(\varepsilon)}(t)}} \le \int_{2b}^{1} \frac{z\Pi(dz)}{\int_0^1 \frac{t(t\wedge(z-b)\Pi^*(dt)}{A_+^{(\varepsilon)}(t)}}$$
$$\le \int_{b}^{1} \frac{2(z-b)\Pi(dz)}{\int_0^1 \frac{t(t\wedge(z-b)\Pi^*(dt)}{A_+^{(\varepsilon)}(t)}} \approx \overline{\mu}_+^{(\varepsilon)}(b).$$

Using $A_+^{(\varepsilon)}(t) \le t\overline{\mu}_+^{(\varepsilon)}(0+)$ gives

$$\int_0^{2b} \frac{z\Pi(dz)}{\int_0^1 \frac{t(t\wedge z)\Pi^*(dt)}{A_+^{(\varepsilon)}(t)}} \le \overline{\mu}_+^{(\varepsilon)}(0+) \int_0^{2b} \frac{z\Pi(dz)}{\int_0^1 (t\wedge z)\Pi^*(dt)}$$
$$= \overline{\mu}_+^{(\varepsilon)}(0+)I(2b), \text{ where } I(x) = \int_0^x \frac{z\Pi(dz)}{\int_0^z \overline{\Pi}_-(t)dt}.$$

Consequently

$$\overline{\mu}_+^{(\varepsilon)}(0+) \le C\{\overline{\mu}_+^{(\varepsilon)}(b) + \overline{\mu}_+^{(\varepsilon)}(0+)I(2b)\},$$

where C does not depend on ε. As $\varepsilon \downarrow 0$ we have $\overline{\mu}_+^{(\varepsilon)}(b) \to \overline{\mu}_+(b) < \infty$ and $\overline{\mu}_+^{(\varepsilon)}(0+) \to \overline{\mu}_+(0+) = \infty$, and we conclude that $I(2b) \ge 1/C > 0$ for all b, which contradicts $I < \infty$, and the result follows. ■

We mention that we can deduce an apparently stronger statement, viz

Corollary 11. *Whenever $X^{(\pm)}$ are independent driftless subordinators, with Lévy measures Π and Π^*, we have*

$$\limsup_{t\downarrow 0} \frac{X_t^{(+)}}{X_t^{(-)}} = 0 \text{ or } \infty \text{ a.s.}$$

according as I is finite or infinite.

This follows by applying Theorem 22 to $X_t^{(+)} - aX_t^{(-)}$. It should also be noted that when the limsup is ∞, it is actually the case that

$$\limsup_{t\downarrow 0} \frac{\Delta_t^{(+)}}{X_t^{(-)}} = \infty \text{ a.s.},$$

where $\Delta^{(+)}$ denotes the jump process of $X^{(+)}$. Finally Vigon [102] shows that I being finite is sufficient for the limsup to be 0, even when the subordinators are dependent; by specialising to the case where they are the ladder time and ladder height processes of some Lévy process Y, he deduces a necessary and sufficient condition for

$$\liminf \frac{\sup_{s\le t} Y_s}{f(t)} = 0 \text{ or } \infty \text{ a.s.};$$

where f is a positive subadditive function.

6.7 Summary: Four Integral Tests

(i) Erickson's test says that a NASC for $X_t \stackrel{a.s.}{\to} -\infty$ as $t \to \infty$ is

$$\begin{aligned}&\mathbb{E}X_1^+ < \infty, \mathbb{E}X_1 < 0, \text{ or}\\ &\mathbb{E}X_1^+ = \mathbb{E}X_1^- = \infty, \text{ and } \int_1^\infty \frac{x\Pi(dx)}{\int_0^x \overline{\Pi^*}(y)dy} < \infty.\end{aligned}$$

Note that $X_t \stackrel{a.s.}{\to} -\infty$ as $t \to \infty$ is equivalent to the existence of $t_0(\omega) < \infty$ such that $X_t < 0$ for all $t > t_0(\omega)$.

(ii) Bertoin's test says that a NASC for 0 to be irregular for $(0, \infty)$ is

$$X \text{ has bounded variation and either its drift } \delta < 0 \text{ or}$$
$$\delta = 0 \text{ and} \int_0^1 \frac{x\Pi(dx)}{\int_0^x \overline{\Pi^*}(y)dy} < \infty.$$

Note that 0 being irregular for $(0, \infty)$ is equivalent to the existence of $t_0(\omega) > 0$ such that $X_t < 0$ for all $0 < t < t_0(\omega)$.

(iii) Chow's test says that a NASC for the mean of the ladder height process, $\mathbb{E}H_1^+$, to be finite is

$$\mathbb{E}|X_1| < \infty \text{ and either } \mathbb{E}X_1 \in (0, \infty), \text{ or } \mathbb{E}X_1 = 0 \text{ and}$$
$$\int_1^\infty \frac{x\overline{\Pi}(x)dx}{\int_0^x \overline{\overline{\Pi^*}}(y)dy} < \infty.$$

Note that $\mathbb{E}H_1^+ < \infty$ is equivalent to $x^{-1}O_x \stackrel{a.s.}{\to} 0$ as $x \to \infty$, where $O_x = X(T_x) - x$ is the overshoot over level x.

(iv) Vigon's test says that a NASC for δ_+, the drift of the ladder height process H_+, to be positive, is

$$\sigma^2 > 0, \text{ or } \sigma^2 = 0 \text{ and either}$$
$$X \text{ has bounded variation with } \delta > 0, \text{ or}$$
$$X \text{ has infinite variation and } \int_0^1 \frac{x\overline{\Pi}_1(x)dx}{\int_0^x \overline{\overline{\Pi^*}}_1(y)dy} < \infty.$$

(Here $\overline{\Pi}_1(x) = \Pi((x, 1))$ etc.) Note that $\delta^+ > 0$ is equivalent to $x^{-1}O_x \stackrel{a.s.}{\to} 0$ as $x \downarrow 0$, and also to X creeping upwards.

7

Spitzer's Condition

7.1 Introduction

We have seen that Spitzer's condition

$$\frac{1}{t}\int_0^t \mathbb{P}\{X_s > 0\}ds \to \rho \in (0,1) \text{ as } t \to \infty \textbf{ or } \text{as } t \to 0+ \tag{7.1.1}$$

is important, essentially because it is equivalent to the ladder time subordinators being asymptotically stable, and hence to the Arc-sine laws holding. Obviously (7.1.1) is implied by

$$\mathbb{P}\{X_t > 0\} \to \rho, \tag{7.1.2}$$

and in 40 years no-one was able to give an example of (7.1.1) holding and (7.1.2) failing, either in the Lévy process or random walk context. What we will see is that they are in fact equivalent, and this equivalence also extends to the degenerate cases $\rho = 0, 1$.

Theorem 23. *For any Lévy process X and for any $0 \le \rho \le 1$, the statements* (7.1.1) *and* (7.1.2) *are equivalent (as $t \to \infty$, or as $t \to 0+$).*

Since the case $t \to \infty$ can be deduced from the random walk results in Doney [33], we will deal here with the case $t \to 0+$. Following Bertoin and Doney [18], we treat the case $\rho = 0, 1$, first, and then give two different proofs for $0 < \rho < 1$. The first is the simplest; it is based on a duality identity for the ladder time processes and does not use any local limit theorem. The second is essentially an adaptation of my method for random walks; in particular it requires a version of the local limit theorem for small times, and a Wiener–Hopf result from Chapter 5.

7.2 Proofs

The purpose of this section is to prove Theorem 23 when $t \to 0+$. The case when the Lévy process $X = (X_t, t \ge 0)$ is a compound Poisson process with drift is of no interest, since in this case $\rho(t) \to 0$ or 1 according as the drift is positive or non-positive, so we will exclude this case. It then follows that $\mathbb{P}\{X_t = 0\} = 0$ for all $t > 0$, and that the mapping $t \to \rho(t) = \mathbb{P}\{X_t > 0\}$ is continuous on $(0, \infty)$ (because X is continuous in probability).

7.2.1 The Case $\rho = 0, 1$

The argument relies on a simple measure-theoretic fact.

Lemma 6. *Let $B \subset [0, \infty)$ be measurable set such that*

$$\lim_{t \to 0+} t^{-1} m(B \cap [0, t]) = 1,$$

where m denotes Lebesgue measure. Then $B + B \supset (0, \varepsilon)$ for some $\varepsilon > 0$.

Proof. Pick $c > 0$ such that $t^{-1} m(B \cap [0, t]) > 3/4$ for all $t \leq c$. Then

$$m(B \cap [t, 2t]) \geq \frac{1}{2} t \quad \text{for all} \quad t < \frac{1}{2} c. \tag{7.2.1}$$

Suppose now that there exists $t < \frac{1}{2}c$ such that $2t \notin B + B$. Then for every $s \in [0, t] \cap B$, $2t - s \in B^c \cap [t, 2t]$ and therefore

$$\begin{aligned} m(B \cap [t, 2t]) &= t - m(B^c \cap [t, 2t]) \\ &\leq t - m(2t - B \cap [0, t]) \\ &\leq t - m(B \cap [0, t]) < \frac{1}{4} t, \end{aligned}$$

and this contradicts (7.2.1). ∎

We are now able to complete the proof of Theorem 23 (as $t \to 0+$) for $\rho = 0, 1$. Obviously it suffices to consider the case $\rho = 1$, so assume $t^{-1} \int_0^t \rho(s) ds \to 1$, and for $\delta \in (0, 1)$ consider $B = \{t : \rho(t) \geq \delta\}$. Then B satisfies the hypothesis of Lemma 6 and we have that $B + B \supset (0, \varepsilon)$ for some $\varepsilon > 0$. For any $t \in (0, \varepsilon)$ choose $s \in (0, t) \cap B$ with $t - s \in B$, so that $\rho(s) \geq \delta$ and $\rho(t - s) \geq \delta$. Then by the Markov property

$$\rho(t) = \mathbb{P}\{X_t > 0\} \geq \mathbb{P}\{X_s > 0\} \mathbb{P}\{X_{t-s} > 0\} \geq \delta^2.$$

Since δ can be chosen arbitrarily close to 1, we conclude that $\lim_{t \to 0+} \rho(t) = 1$.

7.2.2 A First Proof for the Case $0 < \rho < 1$

Recall that the ladder time subordinator $\tau = L^{-1}$ is the inverse local time at the supremum, and has Laplace exponent

$$\Phi(q) = \exp \left\{ \int_0^\infty \left(e^{-t} - e^{-qt} \right) t^{-1} \rho(t) dt \right\}, \qquad q \geq 0. \tag{7.2.2}$$

Also from Corollary 3 in Chapter 4 we know that, with an appropriate choice of the normalisation of local time, the Laplace exponent Φ^* corresponding to the dual Lévy process $X^* = -X$ satisfies

$$\Phi(q)\Phi^*(q) = q.$$

So differentiating (7.2.2) we see that

$$\int_0^\infty e^{-qt}\rho(t)dt = \Phi'(q)/\Phi(q) = \Phi'(q)\Phi^*(q)/q\,. \tag{7.2.3}$$

Suppose now that (7.1.1) holds as $t \to 0+$. By results discussed in Chapter 2, this implies that Φ is regularly varying at ∞ with index ρ, and hence also that Φ^* is regularly varying at ∞ with index $1-\rho$. Because Φ and Φ^* are Laplace exponents of subordinators with zero drift, we obtain from the Lévy–Khintchine formula that

$$\Phi'(q) = \int_0^\infty e^{-qx}xd\left(-T(x)\right), \quad \Phi^*(q)/q = \int_0^\infty e^{-qx}T^*(x)dx\,,$$

where T (respectively, T^*) is the tail of the Lévy measure of the ladder time process of X (respectively, of X^*). We now get from (7.2.3)

$$\rho(t) = \int_{(0,t)} T^*(t-s)sd\left(-T(s)\right) \qquad \text{for a.e. } t > 0\,. \tag{7.2.4}$$

By a change of variables, the right-hand-side can be re-written as

$$t\int_{(0,1)} T^*(t(1-u))ud\left(-T(tu)\right) = \int_{(0,1)} \frac{T^*(t(1-u))}{\Phi^*(1/t)}ud\left(-\frac{T(tu)}{\Phi(1/t)}\right).$$

Now, apply a Tauberian theorem, the monotone density theorem and the uniform convergence theorem (see Theorems 1.7.1, 1.7.2 and 1.5.2 in [20]). For every fixed $\varepsilon \in (0,1)$, we have, uniformly on $u \in [\varepsilon, 1-\varepsilon]$ as $t \to 0+$,

$$\frac{T(tu)}{\Phi(1/t)} \to \frac{u^{-\rho}}{\Gamma(1-\rho)}, \quad \frac{T^*(t(1-u))}{\Phi^*(1/t)} \to \frac{(1-u)^{(1-\rho)}}{\Gamma(\rho)}\,.$$

Recall $\rho(t)$ depends continuously on $t > 0$. We deduce from (7.2.4) that

$$\liminf_{t\to 0+} \rho(t) \geq \frac{\rho}{\Gamma(\rho)\Gamma(1-\rho)} \int_\varepsilon^{1-\varepsilon} (1-u)^{\rho-1}u^{-\rho}du\,,$$

and as ε can be picked arbitrarily small, $\liminf_{t\to 0+} \rho(t) \geq \rho$. The same argument for the dual process gives $\liminf_{t\to 0+} \mathbb{P}\{X_t < 0\} \geq 1-\rho$, and this completes the proof.

7.2.3 A Second Proof for the Case $0 < \rho < 1$

Here we will use one of the Wiener–Hopf results we discussed in Chapter 5, specifically

Lemma 7. *We have the following identity between measures on* $(0,\infty) \times (0,\infty)$*:*

$$\mathbb{P}\{X_t \in dx\}dt = t\int_0^\infty \mathbb{P}\{L^{-1}(u) \in dt, H(u) \in dx\}u^{-1}du.$$

We next give a local limit theorem which is more general than we need.

Proposition 10. *Suppose that* $Y = (Y_t, t \geq 0)$ *is a real-valued Lévy process and there exists a measurable function* $r : (0,\infty) \to (0,\infty)$ *such that* $Y_t/r(t)$ *converges in distribution to some law which is not degenerate at a point as* $t \to 0+$*. Then*

(i) *r is regularly varying of index $1/\alpha, 0 < \alpha \leq 2$, and the limit distribution is strictly stable of index α;*
(ii) *for each $t > 0$, Y_t has an absolutely continuous distribution with continuous density function $p_t(\cdot)$;*
(iii) *uniformly for $x \in \mathbb{R}$, $\lim_{t\to 0+} r(t)p_t(xr(t)) = p^{(\alpha)}(x)$, where $p^{(\alpha)}(\cdot)$ is the continuous density of the limiting stable law.*

Proof. (i) This is proved in exactly the same way as the corresponding result for $t \to \infty$. (ii) If $\Psi(\lambda)$ denotes the characteristic exponent of Y, so that

$$\mathbb{E}(\exp\{i\lambda Y_t\}) = \exp\{-t\Psi(\lambda)\}\,, \qquad t \geq 0, \lambda \in \mathbb{R},$$

then we have $t\Psi(\lambda/r(t)) \to \Psi^{(\alpha)}(\lambda)$ as $t \to 0+$, where $\Psi^{(\alpha)}$ is the characteristic exponent of a strictly stable law of index α. Because we have excluded the degenerate case, $\mathrm{Re}(\Psi(\lambda))$, the real part of the characteristic exponent (which is an even function of λ), is regularly varying of index α at $+\infty$. It follows that for each $t > 0$, $\exp -t\Psi(\cdot)$ is integrable over $\mathbb{R}$. Consequently (ii) follows by Fourier inversion, which also gives

$$r(t)p_t(xr(t)) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp -\{i\lambda x + t\Psi(\lambda/r(t))\}d\lambda$$

and

$$p^{(\alpha)}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp -\{i\lambda x + \Psi^{(\alpha)}(\lambda)\}d\lambda.$$

(iii) In view of the above formulae, it suffices to show that

$$|\exp -t\Psi(\lambda/r(t))| = \exp -tRe\Psi(\lambda/r(t))$$

is dominated by an integrable function on $|\lambda| \geq K$ for some $K < \infty$ and all small enough λ. But this follows easily from Potter's bounds for regularly varying functions. (See [20], Theorem 1.5.6.) ■

We assume from now on that (7.1.1) holds as $t \to 0+$, so that $\Phi(\lambda)$, the Laplace exponent of the subordinator τ, is regularly varying at ∞ with index ρ. It follows that if we denote by a the inverse function of $1/\Phi(1/\cdot)$, then a is regularly varying with index $1/\rho$ and $\tau(t)/a(t)$ converges in distribution to a non-negative stable law of index ρ as $t \to 0+$. In view of Proposition 10, τ_t has a continuous density which we denote by $g_t(\cdot)$, and $a(t)g_t(a(t)\cdot)$ converges uniformly to the continuous stable density, which we denote by $g^{(\rho)}(\cdot)$. Applying Lemma 7, we obtain the following expression for $\rho(t)$ that should be compared with (7.2.4):

$$\rho(t) = t\int_0^\infty g_u(t)u^{-1}du \qquad \text{for a.e. } t > 0\,. \tag{7.2.5}$$

We are now able to give an alternative proof of Theorem 23 for $0 < \rho < 1$ and $t \to 0+$. By a change of variable,

$$t\int_0^\infty g_u(t)u^{-1}du \;=\; t\int_0^\infty g_{su}(t)u^{-1}du,$$

for any $s > 0$. We now choose $s = 1/\Phi(1/t)$, so that $a(s) = t$, and note that

$$tg_{su}(t) = \frac{a(s)}{a(su)} \cdot a(su)g_{su}\left(a(su)\cdot\frac{a(s)}{a(su)}\right).$$

When $t \to 0+$, $s \to 0+$ and since a is regularly varying with index $1/\rho$, $a(s)/a(su)$ converges pointwise to $u^{-1/\rho}$. It then follows from Proposition 10 that

$$\lim_{t\to 0+} tg_{su}(t) = u^{-1/\rho}g^{(\rho)}(u^{-1/\rho}).$$

Recall that $\rho(t)$ depends continuously on $t > 0$, so that (7.2.5) and Fatou's lemma give

$$\liminf_{t\to 0+} \rho(t) \geq \int_0^\infty g^{(\rho)}(u^{-\frac{1}{\rho}})u^{-\frac{1}{\rho}-1}du = \rho\int_0^\infty g^{(\rho)}(v)dv = \rho.$$

Replacing X by $-X$ gives $\limsup_{t\to 0+}\mathbb{P}\{X_t \geq 0\} \leq \rho$, and the result follows.

7.3 Further Results

The ultimate objective is to find a necessary and sufficient condition, in terms of the characteristics of X, for Spitzer's condition to hold. Current knowledge can be summarised as follows.

(i) If X is symmetric it holds with $\rho = 1/2$, both at 0 and ∞.
(ii) If $\sigma \neq 0$ it holds with $\rho = 1/2$ at 0.

(iii) If X is in the domain of attraction of a strictly stable process with positivity parameter ρ either as $t \to \infty$ or as $t \downarrow 0$ it holds correspondingly at ∞ or at 0.
(iv) It holds with $\rho = 1/2$ at ∞ in some situations where X has an almost symmetric distribution, but is not in the domain of attraction of any symmetric stable process: see Doney [28] for the random-walk case.
(v) It holds if Y is strictly stable with positivity parameter ρ and $X = Y(\tau)$ is a subordinated process, τ being an arbitrary independent subordinator; the point here is that τ can be chosen so that X is not in any domain of attraction. (This observation is due to J. Bertoin.)

The only obvious examples where it doesn't hold is in the spectrally one-sided case; this was pointed out in the random-walk case more than 40 years ago by Spitzer! See [94], p. 227.

Again for random walks the only situation where a necessary and sufficient condition is known is the special case $\rho = 1$. This can be extended to the Lévy process case at ∞, the most efficient way of doing this being to use the stochastic bounds from Chapter 4; see Doney [36]. The result there suggests:

Proposition 11. *For any Lévy process X we have $\rho_t = \mathbb{P}(X_t > 0) \to 1$ as $t \to 0$ if and only if $\pi_x := \mathbb{P}(X$ exits $[-x, x]$ at the top$) \to 1$ as $x \to 0$.*

We now have two possible lines of attack: we could try to find the necessary and sufficient condition for $\rho_t \to 1$ directly, and then Proposition 11 says we have also solved the corresponding exit problem; this progamme is carried out in Doney [37]. But instead we will tackle the exit problem, using material from Andrew [6]. We need some notation; we use the functions (all on $x > 0$)

$$N(x) = \Pi((x, \infty)), \quad M(x) = \Pi((-\infty, -x)),$$

$$L(x) = N(x) + M(x), \quad D(x) = N(x) - M(x),$$

$$A(x) = \gamma + D(1) - \int_x^1 D(y)dy = \gamma + \int_{(x,1]} ydD(y) + xD(x),$$

and

$$U(x) = \sigma^2 + 2\int_0^x yL(y)dy.$$

(It might help to observe that $A(x)$ and $U(x)$ are respectively the mean and variance of $\tilde{X}_1^x$, where $\tilde{X}^x$ is the Lévy process we get by replacing each jump in X which is bigger than x, (respectively less than $-x$) by a jump equal to x, (respectively $-x$).)

Note that always $\lim_{x\to 0} U(x) = \sigma^2$ and $\lim_{x\to 0} xA(x) = 0$, and if X is of bounded variation, $\lim_{x\to 0} A(x) = \delta$, the true drift of X. Also we always have $\lim_{x\to\infty} U(x) = VarX_1 \le \infty$ and $\lim_{x\to\infty} x^{-1}A(x) = 0$, and if $\mathbb{E}|X_1| < \infty$, $\lim_{x\to\infty} A(x) = \mathbb{E}X_1$.

In any study of exits from 2-sided intervals the following quantity is of crucial importance:

$$k(x) = x^{-1}|A(x)|+x^{-2}U(x),\ x>0.$$

For Lévy processes, its importance stems from the following bounds, which are due to Pruitt [83], although he uses a function which is slightly different from k.

Let

$$\overline{\overline{X}}(t) = \sup_{0\le s\le t} |X(s)|$$

and write

$$T_r = \inf(t : \overline{\overline{X}}(t) > r\}.$$

Lemma 8. *There are positive constants c_1, c_2, c_3, c_4 such that, for all Lévy processes and all $r>0$, $t>0$,*

$$\mathbb{P}\{\overline{\overline{X}}(t) \ge r\} \le c_1 t k(r),\ \ \mathbb{P}\{\overline{\overline{X}}(t) \le r\} \le \frac{c_2}{tk(r)}, \tag{7.3.1}$$

and

$$\frac{c_3}{k(r)} \le \mathbb{E}(T(r)) \le \frac{c_4}{k(r)}. \tag{7.3.2}$$

Moreover

$$\frac{1}{\lambda^3} \le \frac{k(\lambda x)}{k(x)} \le 3 \text{ for all } x>0 \text{ and } \lambda>1. \tag{7.3.3}$$

Proof of Proposition 11. We start by assuming $\rho_t = \mathbb{P}(X_t > 0) \to 1$ as $t \to 0$, and suppose that $t = l/k(r)$, where $l \in \mathbb{N}$. (Note that with this choice, the bounds in (7.3.1) are $O(1)$.) Take $\tau_0^r = 0$ and for $j = 0, 1, \cdots$ define

$$\tau_{j+1}^r = \inf\{s > \tau_j : |X_s - X_{\tau_j}| > r\}.$$

Suppose now that the event A_j^r occurs for each $0 \le j < l^2$, where

$$A_j^r = \left(\frac{1}{lk(r)} \le \tau_{j+1}^r - \tau_j^r \le \frac{l}{k(r)} \text{ and } X_{\tau_{j+1}^r} \le X_{\tau_j^r} - r\right);$$

then $X_s \le 0$ for $s \in [\tau_1^r, \tau_{l^2}^r]$. Moreover $t = l/k(r) \in [\tau_1^r, \tau_{l^2}^r]$ and

$$\begin{aligned}
\mathbb{P}(X_t \le 0) &\ge \mathbb{P}\left(\bigcap_{j=1}^{l^2} A_j^r\right) = (\mathbb{P}A_1^r)^{l^2} \\
&\ge \left(\left[\mathbb{P}\{X_{\tau_1^r} < 0\} - \mathbb{P}\left\{\tau_1^r > \frac{l}{k(r)}\right\} - \mathbb{P}\left\{\tau_1^r < \frac{1}{lk(r)}\right\}\right]^+\right)^{l^2} \\
&= \left(\left[\mathbb{P}\{X_{T_r} < 0\} - \mathbb{P}\left\{\overline{\overline{X}}\left(\frac{l}{k(r)}\right) \le r\right\} - \mathbb{P}\left\{\overline{\overline{X}}\left(\frac{1}{lk(r)}\right) \ge r\right\}\right]^+\right)^{l^2}.
\end{aligned}$$

Using Lemma 8, we conclude that:

$$\text{when } t = \frac{l}{k(r)}, \ \mathbb{P}(X_t \leq 0) \geq \left(\left[\mathbb{P}\{X_{T_r} < 0\} - \frac{c}{l} \right]^+ \right)^{l^2}. \tag{7.3.4}$$

It is easy to check that $k(r) \to \infty$ as $r \to 0$, unless $X_t \equiv 0$, a case we implicitly exclude. Therefore if we fix l and let $r \downarrow 0$ then $t(r) = l/k(r) \downarrow 0$, so (7.3.4) gives

$$\limsup_{r \downarrow 0} \mathbb{P}\{X_{T_r} < 0\} \leq \frac{c}{l},$$

and the result follows since l is arbitrary. A somewhat similar argument establishes

$$\text{when } t = \frac{l}{k(r)}, \ \mathbb{P}(X_t \geq 0) \geq [\mathbb{P}\{X_{T_r} > 0\}]^{l^2} - \frac{c}{l},$$

which leads quickly to the converse implication, but we omit the details. ■

We will use Lemma 8 in conjunction with the following straight-forward consequence of the compensation formula: let

$$\begin{aligned} U_r(dy) &= \int_0^\infty \mathbb{P}\left\{ \sup_{0 \leq r < t} |X(u)| \leq r, X(t) \in dy \right\} dt \\ &= \int_0^\infty \mathbb{P}\{T_r > t, X(t) \in dy\} dt. \end{aligned}$$

Then:

Lemma 9. *For* $0 \leq |y| \leq r < |z|$ *we have*

$$\mathbb{P}\{X(T(r)-) \in dy, X(T(r)) \in dz\} = U_r(dy) \Pi(dz - y). \tag{7.3.5}$$

In what follows, it is convenient to focus on the situation where $\pi_x \to 0$; of course the results for $\pi_x \to 1$ follow by considering $-X$. It is not difficult to guess that any necessary and sufficient condition for $\pi_x \to 0$ must involve some control over the sizes of the **positive** jumps which occur before T_r, so let us write $\Delta(T_r) = X_{T_r} - X_{T_r -}$ for the jump which takes X out of $[-r, r]$, and

$$\overline{\Delta}(T_r) = \sup\{(\Delta_t)^+ : t \leq T_r\}$$

for the size of the largest positive jump before T_r. Then since

$$\mathbb{E}T_r = \int_{-r}^r U_r(dy),$$

an immediate consequence of Lemma 9 is that for all $r > 0, \delta > 0$

$$N((\delta + 2)r)\mathbb{E}T_r \leq \mathbb{P}\{\Delta_{T_r} > \delta r\} \leq N(\delta r)\mathbb{E}T_r. \tag{7.3.6}$$

Thus, by Lemma 8,

$$\frac{c_3 N((\delta+2)r)}{k(r)} \leq \mathbb{P}\left\{\Delta_{T_r} > \delta r\right\} \leq \frac{c_4 N(\delta r)}{k(r)},$$

and using (7.3.3) we conclude that

$$\frac{(\Delta_{T_r})^+}{r} \xrightarrow{P} 0 \text{ as } r \to 0 \text{ if and only if } \frac{N(r)}{k(r)} \to 0 \text{ as } r \to 0.$$

By another application of the compensation formula we see that

$$\begin{aligned}\mathbb{P}\left\{\overline{\Delta}_{T_r} > \delta r\right\} &= \mathbb{P}\left\{\sum_{0\leq t\leq T_r} \mathbf{1}_{\{\Delta X_t > \delta r\}} \geq 1\right\} \leq \mathbb{E}\left\{\sum_{0\leq t\leq T_r} \mathbf{1}_{\{\Delta X_t > \delta r\}}\right\} \\ &= N(\delta r)\mathbb{E}T_r \leq \frac{c_4 N(\delta r)}{k(r)},\end{aligned}$$

and of course $\mathbb{P}\left\{\overline{\Delta}_{T_r} > \delta r\right\} \geq \mathbb{P}\left\{\Delta_{T_r} > \delta r\right\}$. Finally we see that if $r^{-1}(\Delta_{T_r})^+ \overset{P}{\nrightarrow} 0$, there exists $\delta, \varepsilon > 0$, $r_n \downarrow 0$ with

$$\mathbb{P}\{X(T_{r_n}) > 0\} \geq \mathbb{P}\{\Delta(T_{r_n}) > \varepsilon r_n\} \geq \delta,$$

and since $r + \Delta(T_r) \geq X_{T_r} \geq r$ on $\{X_{T_r} > 0\}$ we see that

$$\mathbb{P}\{\Delta(T_{r_n}) > \frac{\varepsilon}{1+\varepsilon} X(T_{r_n}) > 0\} \geq \mathbb{P}\{\Delta(T_{r_n}) > \varepsilon r_n\} \geq \delta,$$

so that $\overline{\Delta}_{T_r}/X_{T_r} \overset{P}{\nrightarrow} 0$. Since $|X_{T_r}| \geq r$, the reverse implication is obvious, and we have shown the following:

Proposition 12. *The following are equivalent as* $r \downarrow 0$ *:*

$$\textit{(i)}\ \frac{N(r)}{k(r)} \to 0;\ \textit{(ii)}\ \frac{(\Delta_{T_r})^+}{r} \xrightarrow{P} 0;\ \textit{(iii)}\ \frac{\overline{\Delta}_{T_r}}{r} \xrightarrow{P} 0;\ \textit{(iv)}\ \frac{\overline{\Delta}_{T_r}}{X_{T_r}} \xrightarrow{P} 0.$$

Before formulating the final conclusion, we need an intermediate result.

Proposition 13. *A necessary and sufficient condition for* $\pi_x \to 0$ *as* $x \to 0$ *is*

$$\lim_{r\to 0} \frac{N(r)}{k(r)} = 0 \textit{ and } \limsup_{r\to 0} \frac{A(r)}{rk(r)} < 0. \tag{7.3.7}$$

Remark 2. *In the spectrally negative case we have N identically zero, so the first part of (7.3.7) is automatic. It is not difficult to show the second part is actually equivalent to*

$$\sigma = 0 \textit{ and } A(r) \leq 0 \textit{ for all small enough } r. \tag{7.3.8}$$

In particular, in this case $A(r) = \gamma - M(1) + \int_r^1 M(y)dy$*. So when (7.3.8) holds,* $\int_0^1 M(y)dy$ *is finite, and* X *is of bounded variation with drift* $\delta = \gamma - M(1) + \int_0^1 M(y)dy \le 0$*. Thus* $-X$ *is a subordinator, and hence* $\pi_x \equiv 0$*. (In fact, in analogy with later results in Chapter 9, the only possible limits for* π_x *in the case that* X *is spectrally negative and* $-X$ *is not a subordinator lie in* $[1/2, 1]$*.)*

Proof of Proposition 13. We will write $\tilde{\mathbb{P}}^x$ for the measure under which X has the distribution of the truncated process $\tilde{X}^x$ under $\mathbb{P}$, and note that the corresponding Lévy tails are given by

$$\tilde{M}(y) = M(y),\ \tilde{N}(y) = N(y) \text{ for } y < x,$$
$$\tilde{M}(y) = \tilde{N}(y) = 0, \text{ for } y \ge x.$$

As previously observed, $\tilde{\mathbb{E}}^x X_1 = A(x)$, so $X_t - tA(x)$ is a $\tilde{\mathbb{P}}^x-$martingale, and optional stopping gives

$$\tilde{\mathbb{E}}^x X_{T_r} = A(x)\tilde{\mathbb{E}}^x T_r$$

We will work with $x = \lambda r$, and note, from the fact that under $\tilde{\mathbb{P}}^{\lambda r}$ no jumps exceed λr in absolute value, that

$$\begin{aligned}\tilde{\mathbb{E}}^{\lambda r} X_{T_r} &\ge r\tilde{\mathbb{P}}^{\lambda r}\{X_{T_r} > 0\} - (\lambda+1)r\tilde{\mathbb{P}}^{\lambda r}\{X_{T_r} < 0\}\\ &= r - (\lambda+2)r\tilde{\mathbb{P}}^{\lambda r}\{X_{T_r} < 0\},\end{aligned}$$

and

$$\begin{aligned}\tilde{\mathbb{E}}^{\lambda r} X_{T_r} &\le (\lambda+1)r\tilde{\mathbb{P}}^{\lambda r}\{X_{T_r} > 0\} - r\tilde{\mathbb{P}}^{\lambda r}\{X_{T_r} < 0\}\\ &= (\lambda+1)r - (\lambda+2)r\tilde{\mathbb{P}}^{\lambda r}\{X_{T_r} < 0\}.\end{aligned}$$

Thus

$$\frac{1 - r^{-1}A(\lambda r)\tilde{\mathbb{E}}^{\lambda r} X_{T_r}}{(\lambda+2)} \le \tilde{\mathbb{P}}^{\lambda r}\{X_{T_r} < 0\} \le \frac{(\lambda+1) - r^{-1}A(\lambda r)\tilde{\mathbb{E}}^{\lambda r} X_{T_r}}{(\lambda+2)}. \tag{7.3.9}$$

If we now choose $\lambda = 2$ we will have X and $\tilde{X}^{2r}$ agreeing up to time $\tilde{T}_r = T_r$, so this gives

$$\mathbb{P}\{X_{T_r} < 0\} = \tilde{\mathbb{P}}^{2r}\{X_{T_r} < 0\} \le \frac{3}{4} - \frac{r^{-1}A(2r)\mathbb{E}X_{T_r}}{4},$$

and hence, using Lemma 8 again

$$\frac{cA(2r)}{rk(r)} \le \frac{3}{4} - \mathbb{P}\{X_{T_r} < 0\}.$$

Thus

$$\pi_r \to 0 \Longrightarrow \limsup_{r\to 0} \frac{A(r)}{rk(r)} \le -\frac{1}{4}.$$

But also $\pi_r \to 0$ implies $r^{-1}(\Delta_{T_r})^+ \xrightarrow{P} 0$, and by Proposition 12 this implies $\lim_{r\to 0} N(r)/k(r) = 0$. To reverse the argument, we will assume that (7.3.7) holds and prove

$$\lim_{\lambda\to 0} \liminf_{r\to 0} \tilde{\mathbb{P}}^{\lambda r}\{X_{T_r} < 0\} = 1; \tag{7.3.10}$$

then the result follows from

$$\begin{aligned} \lim_{\lambda\to 0} \liminf_{r\to 0} \tilde{\mathbb{P}}^{\lambda r}\{X_{T_r} < 0\} &\le \lim_{\lambda\to 0} \liminf_{r\to 0} \left(\mathbb{P}\{X_{T_r} < 0\} - \mathbb{P}\{\overline{\Delta}_{T_r} \ge \lambda r\}\right) \\ &\le \liminf_{r\to 0} \mathbb{P}\{X_{T_r} < 0\}, \end{aligned}$$

where we have used Proposition 12. We do this in two stages; the first step is to deduce from (7.3.9) that $\exists c > 0$ such that

$$\lim_{\lambda\to 0} \liminf_{r\to 0} \tilde{\mathbb{P}}^{\lambda r}\{X_{T_r} < 0\} \ge \frac{1+c}{2}. \tag{7.3.11}$$

By considering the sequence defined by

$$\tau_0 = 0, \tau_{j+1} = \inf\{t > \tau_j : |X_t - X_{\tau_j}| > \lambda r\},$$

it is not difficult to show that for any $r > 0$ and $0 < \lambda < 1/2$

$$\mathbb{E}T_{\lambda r} \le 3\lambda \tilde{\mathbb{E}}^{\lambda r} T_r.$$

Using the left-hand side of (7.3.9) and Lemma 8 gives

$$\tilde{\mathbb{P}}^{\lambda r}\{X_{T_r} < 0\} \ge \frac{1 - \frac{cA(\lambda r)}{\lambda r k(\lambda r)}}{\lambda + 2},$$

and letting $r \to 0$ then $\lambda \to 0$ we get (7.3.11).

Now define $p = (2-c)/4$, where c is the constant in (7.3.11), and denote by $\{S_n, n \ge 0\}$ a simple random walk with $P(S_1 = 1) = p, P(S_1 = -1) = q = 1-p$. Put $\sigma_N = \min\{n : |S_n| > N\}, N \in \mathbb{N}$, so that, since $p < 1/2$, we have $P(S_{\sigma_N} < 0) \to 1$ as $N \to \infty$. Thus given $\varepsilon > 0$ we can choose N, K with $P(S_{\sigma_N} < 0, \sigma_N \le K) \ge 1 - \varepsilon$. Take r and λ sufficiently small so that

$$\tilde{q} := \tilde{\mathbb{P}}^{\lambda r}\{X(T_{r/2N} < 0\} \ge q;$$

then, in the obvious notation

$$\begin{aligned} &\tilde{\mathbb{P}}^{\lambda r}\{X \text{ leaves } [-r/2 + \lambda r K, r/2 + \lambda r K] \text{ downwards}\} \\ &\ge \tilde{P}(S_{\sigma_N} < 0, \sigma_N \le K) \ge P(S_{\sigma_N} < 0, \sigma_N \le K) \ge 1 - \varepsilon. \end{aligned}$$

It follows that

$$\lim_{\lambda\to 0}\liminf_{r\to 0}\tilde{\mathbb{P}}^{\lambda r}\{X \text{ leaves } [-r/3, 2r/3] \text{ downwards}\} \geq 1-\varepsilon,$$

and hence

$$\lim_{\lambda\to 0}\liminf_{r\to 0}\tilde{\mathbb{P}}^{\lambda r}\{X_{T_r} < 0\} \geq (1-\varepsilon)^3.$$

Since ε is arbitrary, (7.3.10) follows. ■

Remark 3. *This proof shows that it is impossible for*

$$-\frac{1}{4} < \limsup_{r\to 0}\frac{A(r)}{rk(r)} < 0$$

to occur; this phenomenom was first observed in the random-walk case in Griffin and McConnell [53].

We can now state our main result.

Theorem 24. *Assume X is not a compound Poisson process: then (i) if $N(0+) > 0$ the following are equivalent;*

$$\pi_x \to 0 \textit{ as } x \to 0; \tag{7.3.12}$$

$$\rho_t \to 0 \textit{ as } t \to 0; \tag{7.3.13}$$

$$\frac{X_{T_r}}{\overline{\Delta}_{T_r}} \xrightarrow{P} -\infty \textit{ as } r \to 0; \tag{7.3.14}$$

$$\frac{X_t}{\overline{\Delta}_t} \xrightarrow{P} -\infty \textit{ as } t \to 0; \tag{7.3.15}$$

$$\sigma = 0,\ \frac{A(x)}{xN(x)} \to -\infty \textit{ as } x \to 0; \tag{7.3.16}$$

(ii) if $N(0+) = 0$ then (7.3.12)$\Longleftrightarrow$(7.3.13)$\Longleftrightarrow$

$$A(x) \leq 0 \textit{ for all small enough } x. \tag{7.3.17}$$

Proof. (i) First we need the fact that (7.3.16) is equivalent to (7.3.7) from Proposition 13, which we recall is

$$\lim_{x\to 0}\frac{N(x)}{k(x)} = 0 \text{ and } \limsup_{x\to 0}\frac{A(x)}{xk(x)} < 0. \tag{7.3.18}$$

If this holds, clearly

$$\lim_{x\to 0}\frac{A(x)}{xN(x)} = \lim_{x\to 0}\frac{A(x)}{xk(x)}\frac{k(x)}{N(x)} = -\infty,$$

and if $\sigma^2 > 0$ we would have $k(x) \geq \sigma^2/x^2$ and hence

$$\limsup_{x\to 0} \frac{|A(x)|}{xk(x)} \leq \limsup_{x\to 0} x|A(x)| = 0;$$

thus $\sigma = 0$ and (7.3.16) holds. So assume (7.3.16) and note first that

$$\frac{k(x)}{N(x)} = \frac{|A(x)|}{xN(x)} + \frac{U(x)}{x^2N(x)} \geq \frac{|A(x)|}{xN(x)},$$

so $N(x)/k(x) \to 0$. Also

$$\frac{xk(x)}{|A(x)|} = 1 + \frac{U(x)}{x^2k(x)},$$

so since (7.3.16) implies that $A(x) < 0$ for all small x, we see by writing

$$\frac{U(x)}{xA(x)} = \frac{U(x)}{x^2k(x)}\frac{xk(x)}{A(x)}$$

that

$$\limsup_{x\to 0} \frac{A(x)}{xk(x)} < 0 \text{ if and only if } \liminf_{x\to 0} \frac{U(x)}{xA(x)} > -\infty.$$

Now given $\varepsilon > 0$ we have $yN(y) \leq -\varepsilon A(y)$ for all $y \leq x_0$. Also integration by parts gives

$$\int_0^x A(y)dy = xA(x) - \int_0^x yN(y)dy + \int_0^x yM(y)dy.$$

So for $x \leq x_0$

$$\int_0^x yN(y)dy \leq -\varepsilon xA(x) + \varepsilon \int_0^x yN(y)dy - \varepsilon \int_0^x yM(y)dy. \tag{7.3.19}$$

This implies that

$$(1-\varepsilon)\int_0^x yN(y)dy \leq -\varepsilon xA(x),$$

and also, putting $\varepsilon = 1$ in (7.3.19), that $\int_0^x yM(y)dy \leq -xA(x)$. Thus

$$U(x) = 2\int_0^x y(N(y)+M(y))dy \leq -xA(x)\frac{2\varepsilon}{1-\varepsilon},$$

for all $x \le x_0$, and the result (7.3.18) follows. The equivalence of (7.3.12), (7.3.13), (7.3.14) and (7.3.16) now follows from Propositions 11, 12, and 13, bearing in mind that

$$\pi_x \to 0 \text{ and } \frac{\Delta_{T_r}}{X_{T_r}} \xrightarrow{P} 0 \Longrightarrow \frac{X_{T_r}}{\Delta_{T_r}} \xrightarrow{P} -\infty.$$

Since (7.3.15) obviously implies (7.3.13), we are left to prove that

$$\mathbb{P}\{X_t < 0\} \to 1 \Longrightarrow \frac{X_t}{\overline{\Delta}_t} \xrightarrow{P} -\infty \text{ as } t \to 0.$$

The argument here proceeds by contradiction; so assume $\exists\, t_j \downarrow 0$ with $\mathbb{P}C_j \ge 8\varepsilon > 0$ for all j, where $C_j = \{X_{t_j} > -2k\overline{\Delta}_{t_j}\}$ and k is a fixed integer. Then for each j we can choose c_j such that

$$\mathbb{P}\{(\overline{\Delta}_{t_j} \le c_j) \cap C_j\} \ge 2\varepsilon \text{ and } \mathbb{P}\{(\overline{\Delta}_{t_j} \ge c_j) \cap C_j\} \ge 6\varepsilon. \tag{7.3.20}$$

It follows that for each j at least one of the following must hold:

$$\mathbb{P}\{(\overline{\Delta}_{t_j} > 2c_j) \cap C_j\} \ge 2\varepsilon \tag{7.3.21}$$

or

$$\mathbb{P}\{(c_j \le \overline{\Delta}_{t_j} \le 2c_j) \cap C_j\} \ge 4\varepsilon. \tag{7.3.22}$$

Suppose (7.3.21) holds for infinitely many j. Then write N_t^j for the number of jumps exceeding $2c_j$ which occur by time t, Z_t^j for the sum of these jumps, and $Y_t^j = X_t - Z_t^j$. Of course $N_{t_j}^j$ has a Poisson distribution, and we denote its parameter by p_j. Note that we have

$$\mathbb{P}\{N_{t_j}^j = 0\} \ge \mathbb{P}\{(\overline{\Delta}_{t_j} \le c_j) \cap C_j\} \ge 2\varepsilon \text{ and}$$

$$\mathbb{P}\{N_{t_j}^j > 0\} \ge \mathbb{P}\{(\overline{\Delta}_{t_j} > 2c_j) \cap C_j\} \ge 2\varepsilon,$$

so p_j is bounded uniformly away from 0 and ∞. It follows that $\exists \nu > 0$ with

$$\mathbb{P}\{N_{t_j}^j \ge k\} > e^{-p_j}\frac{p_j^k}{k!} > \nu \text{ for all } j.$$

Also

$$\mathbb{P}\{Z_{t_j}^j = 0, Y_{t_j}^j \in (-2kc_j, 0)\} \ge \mathbb{P}\{C_j \cap (X_{t_j} < 0) \cap (\overline{\Delta}_{t_j} \le c_j)\} \ge \varepsilon$$

for all large j, by (7.3.20) and the fact that $\mathbb{P}(X_{t_j} < 0) \to 1$. So, as Y and Z are independent, the contradiction follows from

$$\liminf_{j\to\infty} \mathbb{P}(X_{t_j} > 0) \ge \liminf_{j\to\infty} \mathbb{P}\{N_{t_j}^j \ge k, Y_{t_j}^j \in (-2kc_j, 0)\} \ge \nu\varepsilon.$$

The second case, when (7.3.22) holds for infinitely many j, can be dealt with in a similar way; see [6] for the details.

(ii) This follows from Propositions 11 and 13, and Remark 2. ■

Some comments on this result are in order.

- The condition (7.3.16) can be shown to be equivalent to

$$\frac{A(x)}{\sqrt{U(x)N(x)}} \to -\infty. \tag{7.3.23}$$

- There are other conditions we can add to the equivalences in Theorem 24. In particular,

$$\exists \text{ a slowly varying } l \text{ such that } \frac{X_t}{tl(t)} \xrightarrow{P} -\infty. \tag{7.3.24}$$

  (This is demonstrated in [37].) Note that this implies $t^{-\alpha}X_t \xrightarrow{P} -\infty$ for any $\alpha > 1$.

- At the cost of considerable extra work, it is possible to give analogous results for sequential limits; see Andrew [6] for the Lévy-process case and Kesten and Maller [62] for the random-walk case.
- Remarkably, the equivalences stated in Theorem 24, and their equivalence to (7.3.23) and (7.3.24), remain valid if limits at zero are replaced by limits at infinity throughout, with only one exception: the large time version of (7.3.16) places no restriction on σ, since the Brownian component is irrelevant for large t. One further difference is that one can add one further equivalence in the $t \to \infty$ case, which is

$$X_t \xrightarrow{P} -\infty \text{ as } t \to \infty.$$

- Suppose X is spectrally positive, so that

$$\frac{A(x)}{xN(x)} = \frac{\gamma + N(1) - \int_x^1 N(y)dy}{xN(x)}.$$

  If X is of bounded variation, i.e. $\int_0^1 N(y)dy < \infty$, then $xN(x) \to 0$ and (7.3.16) is equivalent to $d = \gamma + N(1) - \int_0^1 N(y)dy < 0$. Otherwise, it is equivalent to

$$\frac{\int_x^1 N(y)dy}{xN(x)} \to \infty,$$

  and this happens if and only if $\int_x^1 N(y)dy$ is slowly varying, so that X is "almost" of bounded variation. Note also that a variation of the above shows that in all cases $\int_x^1 N(y)dy$ being slowly varying is **necessary** in order that (7.3.16) holds; of course this includes the case $\int_0^1 N(y)dy < \infty$.

7.4 Tailpiece

None of this helps in finding the necessary and sufficient condition for Spitzer's condition when $0 < \rho < 1$; if anything it suggests how difficult this problem is. This is reinforced by the following results, taken from Andrew [7].

(i) Given any $0 < \alpha \leq \beta < 1$ there are Lévy processes with

$$\alpha = \liminf \pi_x, \ \beta = \limsup \pi_x,$$

and other Lévy processes with

$$\alpha = \liminf \rho_t, \ \beta = \limsup \rho_t.$$

(ii) For any $0 < \alpha < 1$ there is a Lévy process with

$$\alpha = \lim \pi_x = \lim \rho_t.$$

(Non-symmetric stable processes are examples where the two limits exist, but differ.)

(iii) For any $0 < \alpha < \beta < 1$ there is a Lévy process with $\alpha = \lim \rho_t$ and such that π_x fluctuates between α and β for small x.

In conclusion; **every** type of limit behaviour seems to be **possible**.

8

Lévy Processes Conditioned to Stay Positive

8.1 Introduction

In the theory of real-valued diffusions, the concept of "conditioning to stay positive" has proved quite fruitful, in particular in the Brownian case. The basic idea is to find an appropriate function which is invariant (i.e. harmonic) for the process killed on leaving the positive half-line, and then use Doob's h-transform technique. In this chapter we investigate how these ideas can be applied to Lévy processes. It should be mentioned that the first investigations of this question were devoted to the special case where the Lévy process is spectrally one-sided, (see Bertoin, [10] and Chapter VII of [12]), but we will deal with the general case, basically following Chaumont [24] and Chaumont and Doney [25].

8.2 Notation and Preliminaries

Note that the state 0 is regular for $(-\infty, 0)$ under $\mathbb{P}$ if and only if it is regular for $\{0\}$ for the reflected process. In this case, we will simply say that 0 is regular downwards and if 0 is regular for $(0, \infty)$ under $\mathbb{P}$, we will say that 0 is regular upwards. **We will assume that 0 is regular downwards throughout this chapter**. (But see remark 4; also note this precludes the possibility that X is compound Poisson.)

We write T_A for the entrance time into a Borel set A, and m for the time at which the absolute infimum is attained:

$$T_A = \inf\{s > 0 : X_s \in A\}, \tag{8.2.1}$$

$$m = \sup\{s < \zeta : X_s \wedge X_{s-} = \underline{X}_s\}, \tag{8.2.2}$$

where $\underline{X}_s = \inf_{u \le s} X_u$. Let $\underline{L}$ be the local time of the reflected process $X - \underline{X}$ at 0 and let $\underline{n}$ be the characteristic measure of its excursions away from 0. Because of our assumption, $\underline{L}$ is continuous.

Let us first consider the function h defined for all $x \ge 0$ by:

$$h(x) := \mathbb{E}\left(\int_{[0,\infty)} \mathbb{1}_{\{\underline{X}_t \ge -x\}}\, d\underline{L}_t\right). \tag{8.2.3}$$

Making the obvious change of variable we see that

$$h(x) := \mathbb{E}\left(\int_{[0,\infty)} \mathbb{1}_{\{H^*_s \le x\}}\, ds\right)$$

is also the renewal function in the downgoing ladder height process H^*.

It follows from (8.2.3) (or (8.2.5) below) and general properties of Lévy processes that h is *finite, continuous, increasing, and subadditive* on $[0,\infty)$, and that $h(0) = 0$ (because 0 is regular downwards).

Let $\mathbf{e}_\varepsilon$ be an exponential time with parameter ε, which is independent of $(X, \mathbb{P})$. The following identity can be seen by specialising the argument used to prove Theorem 10 in Chapter 4, or alternatively by appealing to Maisonneuve's exit formula of excursion theory. (See [74].) Let $\eta \ge 0$ denote the drift in the downgoing ladder time process: then for all $\varepsilon > 0$,

$$\begin{aligned}\mathbb{P}_x(T_{(-\infty,0)} > \mathbf{e}_\varepsilon) &= \mathbb{P}(\underline{X}_{\mathbf{e}_\varepsilon} \ge -x)\\ &= \mathbb{E}\left(\int_{[0,\infty)} e^{-\epsilon t}\mathbb{1}_{\{\underline{X}_t \ge -x\}}\, d\underline{L}_t\right)\left[\eta\varepsilon + \underline{n}(\mathbf{e}_\varepsilon < \zeta)\right],\end{aligned} \tag{8.2.4}$$

so that, by monotone convergence, for all $x \ge 0$:

$$h(x) = \lim_{\varepsilon \to 0} \frac{\mathbb{P}_x(T_{(-\infty,0)} > \mathbf{e}_\varepsilon)}{\eta\varepsilon + \underline{n}(\mathbf{e}_\varepsilon < \zeta)}. \tag{8.2.5}$$

In the next lemma we show that, for $x > 0$, h is excessive or invariant for the process $(X, \mathbb{P}_x)$ killed at time $\tau_{(-\infty,0)}$. This result has been proved in the context of potential theory by Silverstein [92] Th. 2, where it is assumed that the semigroup is absolutely continuous, 0 is regular for $(-\infty, 0)$, and $(X, \mathbb{P})$ does not drift to $-\infty$; see also Tanaka [98] Th. 2 and Th. 3. Here, we give a simple proof from [25] based on the representation of h given in (8.2.5). (We point out that in [25] the possibility that $\eta > 0$ was overlooked.) For $x > 0$ we denote by $\mathbb{Q}_x$ the law of the killed process, i.e. for $\Lambda \in \mathcal{F}_t$:

$$\mathbb{Q}_x(\Lambda, t < \zeta) = \mathbb{P}_x(\Lambda,\, t < T_{(-\infty,0)}),$$

and by (q_t) its semigroup.

Lemma 10. *If $(X, \mathbb{P})$ drifts towards $-\infty$ then h is excessive for (q_t), i.e. for all $x \ge 0$ and $t \ge 0$, $\mathbb{E}^{\mathbb{Q}}_x(h(X_t)\mathbb{1}_{\{t<\zeta\}}) \le h(x)$. If $(X, \mathbb{P})$ does not drift to $-\infty$, then h is invariant for (q_t), i.e. for all $x \ge 0$ and $t \ge 0$, $\mathbb{E}^{\mathbb{Q}}_x(h(X_t)\mathbb{1}_{\{t<\zeta\}}) = h(x)$.*

Proof. From (8.2.5), monotone convergence and the Markov property, we have

$$\begin{aligned}
&\mathbb{E}_x^{\mathbb{Q}}(h(X_t)\mathbb{I}_{\{t<\zeta\}})\\
&\quad= \lim_{\varepsilon\to 0}\mathbb{E}_x\left(\frac{\mathbb{P}_{X_t}(T_{(-\infty,0)} > \mathbf{e}_\varepsilon)\mathbb{I}_{\{t\le T_{(-\infty,0)}\}}}{\eta\varepsilon + \underline{n}(\mathbf{e}_\varepsilon < \zeta)}\right)\\
&\quad= \lim_{\varepsilon\to 0}\mathbb{E}_x\left(\frac{\mathbb{I}_{\{T_{(-\infty,0)}>t+\mathbf{e}_\varepsilon\}}}{\eta\varepsilon + \underline{n}(\mathbf{e}_\varepsilon < \zeta)}\right)\\
&\quad= \lim_{\varepsilon\to 0} e^{\varepsilon t}\left(\frac{\mathbb{P}_x(T_{(-\infty,0)} > \mathbf{e}_\varepsilon)}{\eta\varepsilon + \underline{n}(\mathbf{e}_\varepsilon < \zeta)} - \int_0^t \varepsilon e^{-\varepsilon u}\frac{\mathbb{P}_x(T_{(-\infty,0)} > u)}{\eta\varepsilon + \underline{n}(\mathbf{e}_\varepsilon < \zeta)}\,du\right)\\
&\quad= h(x) - \frac{1}{\eta + \underline{n}(\zeta)}\int_0^t \mathbb{P}_x(T_{(-\infty,0)} > u)\,du\,, \qquad (8.2.6)
\end{aligned}$$

where $\underline{n}(\zeta) := \int_0^\infty \underline{n}(\zeta > t)\,dt$. From Proposition 6, Chapter 4, we know that for $x > 0$, $\mathbb{E}_x(T_{(-\infty,0)}) < \infty$ if and only if X drifts towards $-\infty$. Hence, since moreover $0 < h(x) < +\infty$ for $x > 0$, then (8.2.5) shows that $\underline{n}(\zeta) < +\infty$ if and only if X drifts towards $-\infty$. Consequently, from (8.2.6), if X drifts towards $-\infty$, then $\mathbb{E}_x^{\mathbb{Q}}(h(X_t)\mathbb{I}_{\{t<\zeta\}}) \le h(x)$, for all $t \ge 0$ and $x \ge 0$, whereas if $(X,\mathbb{P})$ does not drift to $-\infty$, then $\underline{n}(\zeta) = +\infty$ and (8.2.6) shows that $\mathbb{E}_x^{\mathbb{Q}}(h(X_t)\mathbb{I}_{\{t<\zeta\}}) = h(x)$, for all $t \ge 0$ and $x \ge 0$. ∎

8.3 Definition and Path Decomposition

We now define the Lévy process $(X,\mathbb{P}_x)$ conditioned to stay positive. This notion now has a long history; see Bertoin [11], Chaumont [23] and [24], Duquesne [44], Tanaka [98], and the references contained in those papers.

Write $(p_t, t \ge 0)$ for the semigroup of $(X,\mathbb{P})$ and recall that $(q_t, t \ge 0)$ is the semigroup of the process $(X,\mathbb{Q}_x)$. Then we introduce the new semigroup

$$p_t^{\uparrow}(x,dy) := \frac{h(y)}{h(x)}q_t(x,dy),\ \ x > 0, y > 0,\ \ t \ge 0\,. \qquad (8.3.1)$$

From Lemma 10, $(p_t^{\uparrow})$ is sub-Markov when $(X,\mathbb{P})$ drifts towards $-\infty$ and it is Markov in the other cases. For $x > 0$ we denote by $\mathbb{P}_x^{\uparrow}$ the law of the strong Markov process started at x and whose semigroup in $(0,\infty)$ is $(p_t^{\uparrow})$. When $(p_t^{\uparrow})$ is sub-Markov, $(X,\mathbb{P}_x^{\uparrow})$ has state space $(0,\infty)\cup\{\delta\}$ and this process has finite lifetime. In all cases, for $\Lambda \in \mathcal{F}_t$, we have

$$\mathbb{P}_x^{\uparrow}(\Lambda, t < \zeta) = \frac{1}{h(x)}\mathbb{E}_x^{\mathbb{Q}}(h(X_t)\mathbb{I}_\Lambda\mathbb{I}_{\{t<\zeta\}})\,. \qquad (8.3.2)$$

We show in the next proposition that $\mathbb{P}_x^{\uparrow}$ is the limit as $\varepsilon \downarrow 0$ of the law of the process under $\mathbb{P}_x$ conditioned to stay positive up to an independent exponential time with parameter ε, so we will refer to $(X,\mathbb{P}_x^{\uparrow})$ as the process "conditioned to stay positive". Note that the following result has been proved in Th. 1 of [24] under the same assumptions that Silverstein [92] required for his Th. 2, but here we only assume that 0 is regular downwards.

Proposition 14. *Let* $\mathbf{e}_\varepsilon$ *be an exponential time with parameter* ε *which is independent of* $(X, \mathbb{P})$.

For any $x > 0$, *and any* $(\mathcal{F}_t)$ *stopping time* T *and for all* $\Lambda \in \mathcal{F}_T$,

$$\lim_{\varepsilon \to 0} \mathbb{P}_x(\Lambda, T < \mathbf{e}_\varepsilon \,|\, X_s > 0,\, 0 \leq s \leq \mathbf{e}_\varepsilon) = \mathbb{P}_x^\uparrow(\Lambda, T < \zeta)\,.$$

Proof. According to the Markov property and the lack-of-memory property of the exponential law, we have

$$\mathbb{P}_x(\Lambda,\, T < \mathbf{e}_\varepsilon \,|\, X_s > 0,\, 0 \leq s \leq \mathbf{e}_\varepsilon) = \mathbb{E}_x\left(\mathbb{I}_\Lambda \mathbb{I}_{\{T < \mathbf{e}_\varepsilon \wedge T_{(-\infty,0)}\}} \frac{\mathbb{P}_{X_T}(T_{(-\infty,0)} \geq \mathbf{e}_\varepsilon)}{\mathbb{P}_x(T_{(-\infty,0)} \geq \mathbf{e}_\varepsilon)}\right)\,. \tag{8.3.3}$$

Let $\varepsilon_0 > 0$. From (8.2.3) and (8.2.4), for all $\varepsilon \in (0, \varepsilon_0)$,

$$\mathbb{I}_{\{T < \mathbf{e}_\varepsilon \wedge T_{(-\infty,0)}\}} \frac{\mathbb{P}_{X_T}(T_{(-\infty,0)} \geq \mathbf{e}_\varepsilon)}{\mathbb{P}_x(T_{(-\infty,0)} \geq \mathbf{e}_\varepsilon)} \leq \mathbb{I}_{\{T < T_{(-\infty,0)}\}} \mathbb{E}\left(\int_{[0,\infty)} e^{-\varepsilon_0 t} \mathbb{I}_{\{\underline{X}_t \geq -x\}}\, d\underline{L}_t\right)^{-1} h(X_T)\,, \quad \text{a.s.} \tag{8.3.4}$$

Recall that h is excessive for the semigroup (q_t), hence the inequality of Lemma 10 also holds at any stopping time, i.e. $\mathbb{E}_x^{\mathbb{Q}}(h(X_T)\mathbb{I}_{\{T<\zeta\}}) \leq h(x)$. Since h is finite, the expectation of the right-hand side of (8.3.4) is finite, so that we may apply Lebesgue's theorem of dominated convergence in the right-hand side of (8.3.3) when ε goes to 0. We conclude by using the representation of h in (8.2.5) and the definition of $\mathbb{P}_x^\uparrow$ in (8.3.2). ■

Since 0 is regular downwards, definition (8.3.1) does not make sense for $x = 0$, but in [11] it was shown that in all cases, the law of the process

$$((X - \underline{X})_{g_t + s},\, s \leq t - g_t), \quad \text{where} \quad g_t = sup\{s \leq t : (X - \underline{X})_s = 0\},$$

converges as $t \to \infty$ to a Markovian law under which X starts at 0 and has semigroup $p_t^\uparrow$. (See also Tanaka [98], Th. 7 for a related result.) We will denote this limit law by $\mathbb{P}^\uparrow$, and defer for the moment the obvious question: is $\lim_{x \downarrow 0} \mathbb{P}_x^\uparrow = \mathbb{P}^\uparrow$?

The next theorem describes the decomposition of the process $(X, \mathbb{P}_x^\uparrow)$ at the time of its minimum; it reduces to a famous result due to Williams [103] in the Brownian case. It has been proved under additional hypotheses in [24] Th. 5, in [44] Prop. 4.7, Cor. 4.8, and under the sole assumption that X is not a compound Poisson process in [25].

Theorem 25. *Define the pre-minimum and post-minimum processes respectively as follows:* $(X_t\,,\,0\le t<m)$ *and* $(X_{t+m}-U\,,\,0\le t<\zeta-m)$*, where* $U:=X_m\wedge X_{m-}$*.*

1. *Under* $\mathbb{P}^\uparrow_x$*,* $x>0$*, the pre-minimum and post-minimum processes are independent. The process* $(X,\mathbb{P}^\uparrow_x)$ *reaches its absolute minimum* U *once only and its law is given by:*
$$\mathbb{P}^\uparrow_x(U\ge y)=\frac{h(x-y)}{h(x)}1_{\{y\le x\}}\,. \tag{8.3.5}$$
2. *Under* $\mathbb{P}^\uparrow_x$*, the law of the post-minimum process is* $\mathbb{P}^\uparrow$*. In particular, it is strongly Markov and does not depend on* x*. The semigroup of* $(X,\mathbb{P}^\uparrow)$ *in* $(0,\infty)$ *is* $(p^\uparrow_t)$*. Moreover,* $X_0=0$*,* $\mathbb{P}^\uparrow$*-a.s. if and only if* 0 *is regular upwards.*

Proof. Denote by $\mathbb{P}^{\mathbf{e}_\varepsilon}_x$ the law of the process $(X,\mathbb{P}_x)$ killed at time $\boldsymbol{e}_\varepsilon$. Since $(X,\mathbb{P})$ is not a compound Poisson process, it almost surely reaches its minimum at a unique time on the interval $[0,\mathbf{e}_\varepsilon]$. Recall that by a result in [76], pre-minimum and post-minimum processes are independent under $\mathbb{P}^{\mathbf{e}_\varepsilon}_x$ for all $\varepsilon>0$. According to Proposition 14, the same properties hold under $\mathbb{P}^\uparrow_x$. Let $0\le y\le x$. From Proposition 14 and (8.2.5):

$$\begin{aligned}\mathbb{P}^\uparrow_x(U<y)&=\mathbb{P}^\uparrow_x(T_{[0,y)}<\zeta)=\lim_{\varepsilon\to0}\mathbb{P}_x(T_{[0,y)}<\mathbf{e}_\varepsilon\,|\,T_{(-\infty,0)}>\mathbf{e}_\varepsilon)\\&=\lim_{\varepsilon\to0}\left(1-\frac{\mathbb{P}_x(T_{[0,y)}\ge\mathbf{e}_\varepsilon,\,T_{(-\infty,0)}>\mathbf{e}_\varepsilon)}{\mathbb{P}_x(T_{(-\infty,0)}>\mathbf{e}_\varepsilon)}\right)\\&=1-\lim_{\varepsilon\to0}\frac{\mathbb{P}_{x-y}(T_{(-\infty,0)}\ge\mathbf{e}_\varepsilon)}{\mathbb{P}_x(T_{(-\infty,0)}>\mathbf{e}_\varepsilon)}=1-\frac{h(x-y)}{h(x)}\,,\end{aligned}$$

and the first part of the theorem is proved.

From the independence mentioned above, the law of the post-minimum process under $\mathbb{P}^{\mathbf{e}_\varepsilon}_x(\,\cdot\,|\,U>0)$ is the same as the law of the post-minimum process under $\mathbb{P}^{\mathbf{e}_\varepsilon}_x$. Then, from Proposition 14 or from Bertoin, [11], Corollary 3.2, the law of the post-minimum processes under $\mathbb{P}^\uparrow_x$ is the limit of the law of the post-minimum process under $\mathbb{P}^{\mathbf{e}_\varepsilon}_x$, as $\varepsilon\to0$. But [11], Corollary 3.2, also proved that this limit law is that of a strong Markov process with semigroup $(p^\uparrow_t)$. Moreover, from Millar [77], the process $(X,\mathbb{P}^{\mathbf{e}_\varepsilon}_x)$ leaves its pre-minimum continuously, (that is $\mathbb{P}^{\mathbf{e}_\varepsilon}_x(X_m>X_{m-})=0$) if and only if 0 is regular upwards. Then we conclude the proof of the second statement by using Proposition 14. ∎

Williams' result also contains a description of the pre-minimum process, and Chaumont [24] was able to extend this, under the additional assumption that X has an absolutely continuous semigroup. In this case h has a continuous derivative which satisfies $0<h'(x)<\infty$ for $0<x<\infty$, and h' is also excessive for (q_t). Then, under $\mathbb{P}^\uparrow_x$, the law of the pre-minimum process, conditionally on $X_m = a$, is that of $X + a$ under $\mathbb{P}^{\searrow}_{x-a}$, where $\mathbb{P}^{\searrow}_y$, for $y > 0$, denotes the h' h-transform of $\mathbb{Q}_y$, viz

$$\mathbb{P}^{\searrow}_x(\Lambda, t < \zeta) = \frac{1}{h'(x)} \mathbb{E}^{\mathbb{Q}}_x(h'(X_t) 1\!\!1_\Lambda 1\!\!1_{\{t<\zeta\}}) .$$

Note that in the spectrally positive case, which includes that of Brownian motion, we have $h(x) = x$, so $\mathbb{P}^{\searrow}_y$ is just $\mathbb{Q}_y$. In other cases we can think of $(X, \mathbb{P}^{\searrow}_y)$ as 'X conditioned to die at 0 from above'; see [24], Section 4 for details.

When $(X, \mathbb{P})$ has no negative jumps and 0 is not regular upwards, the initial law of $(X, \mathbb{P}^{\uparrow})$ has been computed in Chaumont [23]. It is given by:

$$\mathbb{P}^{\uparrow}(X_0 \in dx) = \frac{x\,\pi(dx)}{\int_0^\infty u\,\pi(du)}, \quad x \geq 0 , \tag{8.3.6}$$

where π is the Lévy measure of $(X, \mathbb{P})$. It seems difficult to obtain an explicit formula which only involves π in the general case.

8.4 The Convergence Result

For Brownian motion it is easy to demonstrate the weak convergence of $\mathbb{P}^{\uparrow}_x$ to $\mathbb{P}^{\uparrow}$; for a general Lévy process, in view of Theorem 25, this essentially amounts to showing that the pre-minimum process vanishes in probability as $x \downarrow 0$. Such a result has been verified in the case of spectrally negative processes in Bertoin [9], and for stable processes and for processes which creep downwards in Chaumont [24]. For some time this was an open question for other Lévy processes, but in Chaumont and Doney [25] we gave a simple proof of this result for a general Lévy process. This proof does not use the description of the law of the pre-minimum process in Theorem 25 but depends only on knowledge of the distribution of the all-time minimum under $\mathbb{P}^{\uparrow}_x$. In the following, θ_ε is the forward shift operator.

Theorem 26. *Assume that* 0 *is regular upwards. Then the family* $(\mathbb{P}^{\uparrow}_x, x > 0)$ *converges on the Skorokhod space to* $\mathbb{P}^{\uparrow}$. *Moreover the semigroup* $(p^{\uparrow}_t, t \geq 0)$ *satisfies the Feller property on the space* $\mathcal{C}_0([0,\infty))$ *of continuous functions vanishing at infinity.*

If 0 *is not regular upwards, then for any* $\varepsilon > 0$, *the process* $(X \circ \theta_\varepsilon, \mathbb{P}^{\uparrow}_x)$ *converges weakly towards* $(X \circ \theta_\varepsilon, \mathbb{P}^{\uparrow})$, *as* x *tends to* 0.

Proof. Let $(\Omega, \mathcal{F}, P)$ be a probability space on which we can define a family of processes $(Y^{(x)})_{x>0}$ such that each process $Y^{(x)}$ has law $\mathbb{P}^{\uparrow}_x$. Let also Z be a process with law $\mathbb{P}^{\uparrow}$ which is independent of the family $(Y^{(x)})$. Let m_x be the unique hitting time of the minimum of $Y^{(x)}$ and define, for all $x > 0$, the process $Z^{(x)}$ by:

$$Z_t^{(x)} = \begin{cases} Y_t^{(x)} & t < m_x \\ Z_{t-m_x} + Y_{m_x}^{(x)} & t \geq m_x \,. \end{cases}$$

By Theorem 25, under P, $Z^{(x)}$ has law $\mathbb{P}_x^{\uparrow}$.

Now first assume that 0 is regular upwards, so that $\lim_{t\downarrow 0} Z_t = 0$, almost surely. We are going to show that the family of processes $Z^{(x)}$ converges in probability towards the process Z as $x \downarrow 0$ for the norm of the J_1-Skorohod topology on the space $\mathcal{D}([0,1])$. Let (x_n) be a decreasing sequence of real numbers which tends to 0. For $\omega \in \mathcal{D}([0,1])$, we easily see that the path $Z^{(x_n)}(\omega)$ tends to $Z(\omega)$ as n goes to ∞ in the Skohorod topology, if both $m_{x_n}(\omega)$ and $\overline{Z}_{m_{x_n}}^{(x_n)}(\omega)$ tend to 0. Hence, it suffices to prove that both m_x and $\overline{Z}_{m_x}^{(x)}$ converge in probability to 0 as $x \to 0$. In the canonical notation (i.e. with $(m, \mathbb{P}_x^{\uparrow}) = (m_x, P)$, where m is defined in (10.3.20) and $(X, \mathbb{P}_x^{\uparrow}) = (Z^{(x)}, P)$), we have to show that for any fixed $\varepsilon > 0, \eta > 0$,

$$\lim_{x\downarrow 0} \mathbb{P}_x^{\uparrow}(m > \varepsilon) = 0 \quad \text{and} \quad \lim_{x\downarrow 0} \mathbb{P}_x^{\uparrow}(\overline{X}_m > \eta) = 0. \tag{8.4.1}$$

First, applying the Markov property at time ε gives

$$\begin{aligned} \mathbb{P}_x^{\uparrow}(m > \varepsilon) &= \int_{0<y\leq x} \int_{z>y} \mathbb{P}_x^{\uparrow}(X_\varepsilon \in dz, \underline{X}_\varepsilon \in dy, \varepsilon < \zeta) \mathbb{P}_z^{\uparrow}(U < y) \\ &= \int_{0<y\leq x} \int_{z>y} \mathbb{Q}_x(X_\varepsilon \in dz, \underline{X}_\varepsilon \in dy, \varepsilon < \zeta) \frac{h(z)}{h(x)} \mathbb{P}_z^{\uparrow}(U < y) \\ &= \int_{0<y\leq x} \int_{z>y} \mathbb{P}_x(X_\varepsilon \in dz, \underline{X}_\varepsilon \in dy) \frac{h(z) - h(z-y)}{h(x)}, \end{aligned}$$

where we have used the result of Theorem 25 and the fact that $\mathbb{Q}_x$ and $\mathbb{P}_x$ agree on $\mathcal{F}_\varepsilon \cap (\underline{X}_\varepsilon > 0)$. Since h is increasing and subadditive, we have $h(z) - h(z-y) \leq h(y)$, and so

$$\begin{aligned} \mathbb{P}_x^{\uparrow}(m > \varepsilon) &\leq \frac{1}{h(x)} \int_{0<y\leq x} \int_{z>y} \mathbb{P}_x(X_\varepsilon \in dz, \underline{X}_\varepsilon \in dy) h(y) \\ &= \frac{1}{h(x)} \int_{0<y\leq x} \mathbb{P}_x(\underline{X}_\varepsilon \in dy) h(y) \leq \mathbb{P}_x(\underline{X}_\varepsilon > 0)\,. \end{aligned}$$

Since 0 is regular downwards, we clearly have $\mathbb{P}_x(\underline{X}_\varepsilon > 0) \to 0$ as $x \to 0$, so the result is true.

For the second claim in (8.4.1), we apply the strong Markov property at time $T := T_{(\eta,\infty)}$, with $x < \eta$, to get

$$\begin{aligned} \mathbb{P}_x^{\uparrow}(\overline{X}_m > \eta) &= \int_{z\geq\eta} \int_{0<y\leq x} \mathbb{P}_x^{\uparrow}(X_{\boldsymbol{T}} \in dz, \underline{X}_{\boldsymbol{T}} \in dy, T < \zeta) \mathbb{P}_z^{\uparrow}(U < y) \\ &= \int_{z\geq\eta} \int_{0<y\leq x} \mathbb{P}_x^{\uparrow}(X_{\boldsymbol{T}} \in dz, \underline{X}_{\boldsymbol{T}} \in dy, T < \zeta) \frac{h(z) - h(z-y)}{h(z)}. \end{aligned}$$

We now apply the simple bound

$$\frac{h(z)-h(z-y)}{h(z)} \le \frac{h(y)}{h(z)} \le \frac{h(x)}{h(\eta)} \text{ for } 0<y\le x \text{ and } z\ge\eta$$

to deduce that

$$\mathbb{P}_x^{\uparrow}(\overline{X}_m > \eta) \le \frac{h(x)}{h(\eta)} \to 0 \text{ as } x\downarrow 0\,.$$

Then, the weak convergence of $(\mathbb{P}_x^{\uparrow})$ towards $\mathbb{P}^{\uparrow}$ is proved. When 0 is regular upwards, the Feller property of the semigroup $(p_t^{\uparrow},\, t\ge 0)$ on the space $\mathcal{C}_0([0,\infty))$ follows from its definition in (8.3.1), the properties of Lévy processes and the weak convergence at 0 of $(\mathbb{P}_x^{\uparrow})$.

Finally when 0 is not regular upwards, (8.4.1) still holds but we can check that, at time $t=0$, the family of processes $Z^{(x)}$ does not converge in probability towards 0. However, following the above arguments we can still prove that, for any $\varepsilon>0$, $(Z^{(x)}\circ\theta_\varepsilon)$ converges in probability towards $Z\circ\theta_\varepsilon$ as $x\downarrow 0$. ∎

The following absolute continuity relation between the measure $\underline{n}$ of the process of the excursions away from 0 of $X-\underline{X}$ and $\mathbb{P}^{\uparrow}$ has been established in [24], Th. 3: for $t>0$ and $A\in\mathcal{F}_t$

$$\underline{n}(A, t<\zeta) = k\mathbb{E}^{\uparrow}(h(X_t)^{-1}A), \tag{8.4.2}$$

where $k>0$ is a constant which depends only on the normalization of the local time $\underline{L}$. Relation (8.4.2) was proved in [24] under the additional hypotheses mentioned before Theorem 25 above, but we can easily check that it still holds under the sole assumption that X is not a compound Poisson process. Then a consequence of Theorem 26 is:

Corollary 12. *Assume that* 0 *is regular upwards. For any* $t>0$ *and for any* $\mathcal{F}_t$ *-measurable, continuous and bounded functional* F*,*

$$\underline{n}(F, t<\zeta) = k\lim_{x\to 0}\mathbb{E}_x^{\uparrow}(h(X_t)^{-1}F).$$

Another application of Theorem 26 is to the asymptotic behavior of the semigroup $q_t(x,dy)$, $t>0$, $y>0$, when x goes towards 0. Let us denote by $j_t(dx)$, $t\ge 0$, $x\ge 0$ the the entrance law of the excursion measure $\underline{n}$, that is the Borel function which is defined for any $t\ge 0$ as follows:

$$\underline{n}(f(X_t), t<\zeta) = \int_0^\infty f(x)j_t(dx)\,,$$

where f is any positive or bounded Borel function f.

Corollary 13. *The asymptotic behavior of* $q_t(x,dy)$ *is given by:*

$$\int_0^\infty f(y)q_t(x,dy) \sim_{x\to 0} h(x)\int_0^\infty f(y)j_t(dy)\,,$$

for $t>0$ *and for every continuous and bounded function* f*.*

Remark 4. *In the case that 0 is not regular downwards but X is not compound Poisson most of the results presented so far hold. In this case the set $\{t : (X - \underline{X})_t = 0\}$ is discrete and we define the local time $\underline{L}$ as the counting process of this set, i.e. $\underline{L}$ is a jump process whose jumps have size 1 and occur at each zero of $X - \underline{X}$. Then, the measure $\underline{n}$ is the probability law of the process X under the law $\mathbb{P}$, killed at its first passage time in the negative halfline, i.e. $\tau_{(-\infty,0)}$. We can still define h in the same way, it is still subadditive, but it is no longer continuous and $h(0) = 1$. Lemma 10 remains valid, as do definitions (8.3.1) and (8.3.2), and Proposition 14, which now also make sense for $x = 0$. The decomposition result Theorem 25 also remains valid, as does the convergence result Theorem 26, though its proof requires minor changes.*

8.5 Pathwise Constructions of $(X, \mathbb{P}^{\uparrow})$

In this section we describe two different path constructions of $(X, \mathbb{P}^{\uparrow})$. The first is an extension of a discrete-time result from Tanaka [97], (see also Doney [31]), and the second is contained in Bertoin [11]. These two constructions are quite different from each other but coincide in the Brownian case. Roughly speaking, we could say that the first construction is based on a rearrangement of the excursions away from 0 of the Lévy process reflected at its minimum, whereas Bertoin's construction consists in sticking together the positive excursions away from 0 of the Lévy process itself. In both cases the random-walk analogue is easier to visualise.

8.5.1 Tanaka's Construction

If S is any random walk which starts at zero, has $S_n = \sum_1^n Y_r, n \geq 1$, and does not drift to $-\infty$, we write $S^{\uparrow}$ for the harmonic transform of S killed at time $\sigma := \min(n \geq 1 : S_n \leq 0)$ which corresponds to "conditioning S to stay positive". Thus for $x > 0, y > 0$, and $x = 0$ when $n = 0$

$$\begin{aligned} P(S_{n+1}^{\uparrow} \in dy | S_n^{\uparrow} = x) &= \frac{V^*(y)}{V^*(x)} P(S_{n+1} \in dy | S_n = x) \\ &= \frac{V^*(y)}{V^*(x)} P(S_1 \in dy - x), \end{aligned}$$

where V^* is the renewal function in the weak increasing ladder process of $-S$. In [97] it was shown that a process R got by time-reversing one by one the excursions below the maximum of S has the same distribution as $S^{\uparrow}$; specifically, if $\{(T_k, H_k), k \geq 0\}$ denotes the strict increasing ladder process of S (with $T_0 = H_0 \equiv 0$), then R is defined by

$$R_0 = 0,\ R_n = H_k + \sum_{i=T_{k+1}+T_k+1-n}^{T_{k+1}} Y_i,\ T_k < n \leq T_{k+1},\ k \geq 0. \tag{8.5.1}$$

Thus we can represent R as $[\hat{\delta}_1,\hat{\delta}_2,\cdots]$, where $\hat{\delta}_1,\hat{\delta}_2\cdots$ are the time reversals of the completed excursions below below the maximum of S and $[\cdots]$ denotes concatenation.

To see this, introduce an independent Geometrically distributed random time G_ρ with parameter ρ and put $J_\rho = \max\{n \le G_\rho : S_n = \min_{r\le n} S_r\}$. Then it is not difficult to show that $S^\uparrow$ is the limit, in the sense of convergence of finite-dimensional distributions, of $\tilde{S}_\rho := (S_n, 0 \le n \le G_\rho | \sigma > G_\rho)$ as $\rho \downarrow 0$. (See Bertoin and Doney [17] for a similar result.) On the other hand, it is also easy to verify that $\tilde{S}_\rho$ has the same distribution as the post-minimum process

$$\overrightarrow{S}_\rho := (S_{J_\rho+n} - S_{J_\rho}, 0 \le n \le G_\rho - J_\rho).$$

By time-reversal we see, in the obvious notation, that if K_ρ is the index of the current excursion below the maximum at time G_ρ,

$$\begin{aligned}\overrightarrow{S}_\rho &\overset{D}{=} [\hat{\delta}_{K_\rho}(\rho),\cdots\hat{\delta}_1(\rho)] \\ &\overset{D}{=} [\hat{\delta}_1(\rho),\cdots\hat{\delta}_{K_\rho}(\rho)],\end{aligned} \tag{8.5.2}$$

the second equality following because $\hat{\delta}_1(\rho),\cdots\hat{\delta}_{K_\rho}(\rho)$ are independent and identically distributed and independent of K_ρ. Noting that $\hat{\delta}_1(\rho) \overset{D}{\to} \hat{\delta}_1$ and $K_\rho \overset{a.s.}{\to} \infty$ as $\rho \downarrow 0$, we conclude that $S^\uparrow \overset{D}{=} [\hat{\delta}_1,\hat{\delta}_2,\cdots] \overset{D}{=} R$, which is the required result.

Turning to the Lévy process case, we find a similar description can be deduced from results in the literature. We first note that with $\overline{S}$ denoting the maximum process of the random walk (8.5.1) can be written in the alternative form

$$R_n = \bar{S}_{T_{k+1}} + (\bar{S} - S)_{T_k+T_{k+1}-n},\ T_k < n \le T_{k+1}.$$

Using the usual notation

$$g(t) = \sup(s < t : X_s = \bar{X}_s),\ d(t) = \inf(s > t : X_s = \bar{X}_s),$$

for the left and right endpoints of the excursion of $\bar{X} - X$ away from 0 which contains t, in the Lévy process case we mimic this definition by setting $R_t = \bar{X}_{d(t)} + \tilde{R}_t$, where

$$\tilde{R}_t = \begin{cases} (\bar{X} - X)_{(d(t)+g(t)-t)-} & \text{if } d(t) > g(t), \\ 0 & \text{if } d(t) = g(t). \end{cases}$$

Let $\boldsymbol{e}_\varepsilon$ be an independent $\mathrm{Exp}(\varepsilon)$ random variable and introduce the future infimum process for X killed at time $\boldsymbol{e}_\varepsilon$ by

$$I_t^{(\varepsilon)} = \inf\{X_s : t \le s \le \boldsymbol{e}_\varepsilon\},\ 0 \le t \le \boldsymbol{e}_\varepsilon,$$

and write $I_0^{(\varepsilon)} = X_{J_\varepsilon}$, so that $J_\varepsilon = g(\boldsymbol{e}_\varepsilon)$ is the time at which the infimum of X over $[0, \boldsymbol{e}_\varepsilon)$ is attained. The following result is established in the proof of Lemme 4 in Bertoin [9]; note that, despite the title of that paper, this result is valid for any Lévy process.

Theorem 27. *(Bertoin) Assume that X does not drift to $-\infty$ under $\mathbb{P}$. Then under $\mathbb{P}_0$ the law of $\{(\tilde{R}_t, \bar{X}_{d(t)}), 0 \leq t < J_\varepsilon\}$ coincides with that of*

$$\{((X - I^{(\varepsilon)})_{J_\varepsilon + t}, I^{(\varepsilon)}_{J_\varepsilon + t} - I_0^{(\varepsilon)}), 0 \leq t < \boldsymbol{e}_\varepsilon - J_\varepsilon\}.$$

Of course, an immediate consequence of this is the equality in law of

$$\{R_t, 0 \leq t < J_\varepsilon\} \text{ and } \{X_{J_\varepsilon + t} - I_0^{(\varepsilon)}, 0 \leq t < \boldsymbol{e}_\varepsilon - J_\varepsilon\}.$$

As previously mentioned, as $\varepsilon \downarrow 0$ the distribution of the right-hand side converges to that of $\mathbb{P}^{\uparrow}$ and we conclude that

Theorem 28. *Under $\mathbb{P}_0$ the law of $\{R_t, t \geq 0\}$ is $\mathbb{P}^{\uparrow}$.*

Since the excursions of Brownian motion are invariant under time-reversal, it is easy to deduce, using Pitman's representation (see [82]), that R is Bess(3) in this case.

8.5.2 Bertoin's Construction

For random walks, Bertoin's construction is easy to describe: just remove every step of the walk which takes the walk to a non-positive value. Because we are assuming that S does not drift to $-\infty$, this leaves an infinite number of steps, and the corresponding partial sum process has the law of $S^{\uparrow}$. Notice that this has the effect of juxtaposing the "positive excursions of S away from 0", where we include the initial positive jump but exclude the final negative jump.

Why is this true? The underlying reason is that if we apply this procedure to $S^{(G)} := (S_n, 0 \leq n \leq G)$, where G is constant (or random and independent of S), the resulting process has the same law as the post-minimum process of $S^{(G)}$. This is essentially a combinatorial fact which is implicit in Feller's Lemma; see Lemma 3, Section XII.8 of [47]. Applying this with G as in the previous sub-section and letting $\rho \downarrow 0$ leads to our claim.

For a Lévy process X, a similar prescription works, provided it has no Brownian component; we juxtapose the excursions in $(0, \infty)$ of X away from 0, including the possible initial positive jump across 0 and excluding the possible ultimate negative jump across 0.

Specifically, we introduce the "clocks"

$$A_t^+ = \int_0^t \mathbf{1}_{\{X_s > 0\}} ds, \qquad A_t^- = \int_0^t \mathbf{1}_{\{X_s \leq 0\}} ds,$$

and their right-continuous inverses $\alpha^{\pm}$, so that time substitution by α^+ consists of erasing the non-positive excursions and closing up the gaps. To get the correct behaviour at the endpoints of the excursion intervals, we define $X_t^{\uparrow} = Y^{\uparrow}(\alpha_t^+)$, where

$$Y_t^{\uparrow} = X_t + \sum_{0<s\le t} \{\mathbf{1}_{\{X_s\le 0\}} X_{s-}^+ + \mathbf{1}_{\{X_s>0\}} X_{s-}^-\}. \tag{8.5.3}$$

However if and only if $\sigma \neq 0$, X has a non-trivial semimartingale local time l at 0, which appears in the Meyer–Tanaka formula

$$X_t^+ = \int_0^t \mathbf{1}_{\{X_{s-}>0\}} dX_s + \sum_{0<s\le t} \{\mathbf{1}_{\{X_{s-}\le 0\}} X_s^+ + \mathbf{1}_{\{X_{s-}>0\}} X_s^-\} + \frac{1}{2} l_t;$$

note the left and right limits in the sum are inverted with respect to (8.5.3). In this case (8.5.3) has to be modified by adding the factor $\frac{1}{2} l_t$, which takes account of the local time spent at 0. Although technically more complicated, the proof that $X^{\uparrow}$ has measure $\mathbb{P}^{\uparrow}$ follows the same lines as for the random-walk case, the crucial fact being the identity in law between the post-minimum process and $X^{\uparrow}$ when evaluated for a killed version of X.

If X oscillates, a similar procedure can be applied simultaneously to the negative excursions, to produce a version of $X^{\downarrow}$, i.e. X conditioned to stay negative; furthermore $X^{\uparrow}$ and $X^{\downarrow}$ are independent.

In the Brownian case, the Meyer–Tanaka formula reduces to

$$\begin{aligned} B_{\alpha_t^+} = B_{\alpha_t^+}^+ &= \int_0^{\alpha_t^+} \mathbf{1}_{\{B_{s-}>0\}} dB_s + \frac{1}{2} l_{\alpha_t^+} \\ &= B_t^{(1)} - \inf_{s\le t} \{B_s^{(1)}\}, \end{aligned}$$

where $B^{(1)}$ is a new Brownian motion, and we have used the reflection principle. So we have established the distributional identity

$$\left\{ (B_{\alpha_t^+}, \frac{1}{2} l_{\alpha_t^+}), t \ge 0 \right\} \stackrel{D}{=} \{(B_t - \underline{B}_t, -\underline{B}_t), t \ge 0\},$$

and in this case the construction reduces to

$$B_t^{\uparrow} = B_{\alpha_t^+} + \frac{1}{2} l_{\alpha_t^+} = B_t^{(1)} - 2 \inf_{s\le t} \{B_s^{(1)}\},$$

which is of course the classic decomposition of Bess(3) in Pitman [82].

It is interesting to note that if X is any oscillatory Lévy process the processes $X^{(1)}$, $X^{(2)}$ defined by

$$X_t^{(1)} = \int_0^{\alpha_t^+} \mathbf{1}_{\{X_{s-}>0\}} dX_s, \; X_t^{(2)} = \int_0^{\alpha_t^-} \mathbf{1}_{\{X_{s-}\le 0\}} dX_s$$

are independent copies of X. See Doney [32]. In the case that X is spectrally negative, Bertoin [11] used this observation in establishing a nice extension of Pitman's decomposition. Using similar arguments to those above, he showed the identity

$$\left\{(X_{\alpha_t^+}, \frac{1}{2} l_{\alpha_t^+}, \sum_{0<s\le\alpha_t^+} \mathbf{1}_{\{X_s>0\}} X_{s-}^-), t \ge 0\right\} \stackrel{D}{=}$$
$$\left\{(X_t - \underline{X}_t^{(c)}, -\underline{X}_t^{(c)}, \underline{X}_t^{(c)} - \underline{X}_t), t \ge 0\right\},$$

where $\underline{X}^{(c)}$ denotes the continuous part of the decreasing process $\underline{X}$. As a consequence he was able to establish that if we set

$$\underline{J}_t = \sum_{s\le t} \Delta X_s \mathbf{1}_{\{X_s<\underline{X}_s\}},$$

which is the sum of the jumps across the previous minimum by time t, then the process $X - 2\underline{X}^{(c)} - \underline{J}$ has law $\mathbb{P}^{\uparrow}$.

9

Spectrally Negative Lévy Processes

9.1 Introduction

Spectrally negative Lévy processes form a subclass of Lévy processes for which we can establish many explicit and semi-explicit results, fundamentally because they can only move upwards in a continuous way. Because of this the Wiener–Hopf factors are much more manageable, we can solve the 2-sided exit problem, and the process conditioned to stay positive has some nice properties. It should also be mentioned that an arbitrary Lévy process can be written as the difference of two independent spectrally negative Lévy process, which gives the possibility of establishing general results by studying this subclass of processes.

The main aim of this chapter is to explain some recent developments involving the "generalised scale function", but we start by recalling some basic facts that can be found in Chapter VII of [12].

9.2 Basics

Throughout this Chapter X will be a spectrally negative Lévy process, that is its Lévy measure is supported by $(-\infty, 0)$, so that it has no positive jumps. We will exclude the degenerate cases when X is either a pure drift or the negative of a subordinator, but note our definition includes Brownian motion.

A first consequence of the absence of positive jumps is that the right-hand tail of the distribution of X_t is small; in fact it is not difficult to show that

$$\mathbb{E}(e^{\lambda X_t}) < \infty \text{ for all } \lambda \geq 0. \tag{9.2.1}$$

Thus we are able to work with the Laplace exponent $\psi(\lambda) = -\Psi(-i\lambda)$, which satisfies

$$\mathbb{E}(e^{\lambda X_t}) = \exp\{t\psi(\lambda)\} \text{ for } \mathrm{Re}(\lambda) \geq 0, \tag{9.2.2}$$

and the Lévy–Khintchine formula now takes the form

$$\psi(\lambda) = \gamma\lambda + \frac{1}{2}\sigma^2\lambda^2 + \int_{(-\infty,0)} \left(e^{\lambda x} - 1 - \lambda x \mathbf{1}_{\{x>-1\}}\right) \Pi(dx). \tag{9.2.3}$$

Another consequence of the absence of positive jumps is that for $a \geq 0$ the first-passage time $T[a,\infty)$ satisfies

$$X_{T[a,\infty)} \overset{\text{a.s.}}{=} a \text{ on } \{T[a,\infty) < \infty\}. \tag{9.2.4}$$

From this we deduce that $S_{\boldsymbol{e}(q)}$ has an exponential distribution, with parameter $\Phi(q)$ say, where as usual $\boldsymbol{e}(q)$ denotes an independent random variable with an $\text{Exp}(q)$ distribution, and S is the supremum process. Exploiting (9.2.4), we see that

$$\psi(\Phi(\lambda)) \equiv \lambda, \ \lambda > 0,$$

and since ψ is continuous, eventually increasing and convex, Φ is a bijection: $[0,\infty) \to [\Phi(0),\infty)$. Here $\Phi(0) = 0$ when 0 is the only root of $\psi(\lambda) = 0$, and otherwise it is the larger of the two roots. This leads to the following fundamental result:

Theorem 29. *The point 0 is regular for* $(0,\infty)$ *and the continuous increasing process* S *is a local time at 0 for the reflected process* $S-X$. *Its right-continuous inverse*

$$T_x = \inf\{s \geq 0 : X_s > x\}, \ x \geq 0,$$

is a subordinator, killed at an exponential time if X *drifts to* $-\infty$, *and its Laplace exponent is* Φ.

Of course the killing rate is $\Phi(0)$, and it is clear from a picture that

$$\Phi(0) > 0 \iff \psi'(0+) < 0 \iff \mathbb{E}X_1 < 0 \iff X \text{ drifts to } -\infty,$$

which squares with the fact that

$$\psi'(0+) = 0 \iff \mathbb{E}X_1 = 0 \iff X \text{ oscillates},$$

$$\psi'(0+) > 0 \iff \mathbb{E}X_1 > 0 \iff X \text{ drifts to } \infty.$$

The Wiener–Hopf factorisation now takes the form

$$\frac{q}{q-\psi(\lambda)} = \mathbb{E}(e^{\lambda X_{\boldsymbol{e}(q)}}) = \mathbb{E}(e^{\lambda I_{\boldsymbol{e}(q)}})\mathbb{E}(e^{\lambda S_{\boldsymbol{e}(q)}}), \tag{9.2.5a}$$

with $I_t = \inf_{s \leq t} X_s$, and since we know $\mathbb{E}(e^{\lambda S_{\boldsymbol{e}(q)}}) = \Phi(q)/(\Phi(q)-\lambda)$ we see that the other factor is given by

$$\mathbb{E}(e^{\lambda I_{\boldsymbol{e}(q)}}) = \frac{q(\Phi(q)-\lambda)}{\Phi(q)(q-\psi(\lambda))}, \ \lambda > 0. \tag{9.2.6}$$

A first consequence of this is that when $\mathbb{E}X_1 > 0$ we can let $q \downarrow 0$ to get

$$\mathbb{E}(e^{\lambda I_\infty}) = \frac{\lambda}{\psi(\lambda)\Phi'(0+)} = \frac{\lambda\psi'(0+)}{\psi(\lambda)}, \ \lambda > 0. \tag{9.2.7}$$

Secondly, letting $\lambda \to \infty$ in (9.2.6) we see that $\mathbb{P}(I_{\boldsymbol{e(q)}} = 0) > 0$ if and only if $\lim_{\lambda\to\infty} \lambda^{-1}\psi(\lambda) < \infty$. From the Lévy–Khintchine formula (9.2.3) we see that this happens if and only if $\sigma = 0$ and $\int_0^1 \overline{\Pi^*}(x)dx < \infty$, and this leads to

Proposition 15. *The following are equivalent:*

(i) *0 is irregular for* $\{0\}$*:*

(ii) *0 is irregular for* $(-\infty, 0)$*:*

(iii) $\lim_{\lambda\to\infty} \lambda^{-1}\psi(\lambda) < \infty$:

(vi) X *has bounded variation.*

A further consequence of the fact that $S_{\boldsymbol{e(q)}}$ has an Exp$(\Phi(q))$ distribution comes via the Frullani integral, which gives

$$\begin{aligned}\mathbb{E}(e^{-\lambda S_{\boldsymbol{e(q)}}}) &= \frac{\Phi(q)}{\Phi(q)+\lambda} = \exp\left(\int_0^\infty (e^{-\lambda x}-1)x^{-1}e^{-\Phi(q)x}dx\right) \\ &= \exp\left(\int_0^\infty\int_0^\infty (e^{-\lambda x}-1)x^{-1}e^{-qt}\mathbb{P}(T_x \in dt)dx\right).\end{aligned}$$

On the other hand Fristedt's formula gives

$$\mathbb{E}(e^{-\lambda S_{\boldsymbol{e(q)}}}) = \exp\left(\int_0^\infty\int_0^\infty (e^{-\lambda x}-1)t^{-1}e^{-qt}\mathbb{P}(X_t \in dx)dt\right)$$

and we deduce

Proposition 16. *The measures* $t\mathbb{P}(T_x \in dt)dx$ *and* $x\mathbb{P}(X_t \in dx)dt$ *agree on* $[0,\infty) \times [0,\infty)$.

Another consequence of the absence of positive jumps is that the increasing ladder process H has $H(t) = S(T_t) = t$ on $\{T_t < \infty\}$. It follows that H is a pure drift, killed at rate $\Phi(0)$ if X drifts to $-\infty$. One consequence of this is that we can recognise the previous result as a special case of Proposition 8 in Chapter 5. Another is that, since the increasing ladder time process coincides with $\{T_x, x \geq 0\}$, the bivariate Laplace exponent of the increasing ladder process is given by

$$\kappa(\alpha,\beta) = \Phi(\alpha) + \beta. \tag{9.2.8}$$

This in turn implies that the exponent of the decreasing ladder exponent is given by

$$\kappa^*(\alpha,\beta) = c\frac{\alpha - \psi(\beta)}{\Phi(\alpha) - \beta}, \tag{9.2.9}$$

and in particular the exponent of H^* is $\frac{c\psi(\beta)}{\beta - \Phi(0)}$.

We finish this section by introducing the **exponential family** associated with X. It is obvious that for any c such that $\psi(c)$ is finite we can define a measure under which X is again a spectrally negative Lévy process and has exponent $\psi(\lambda + c) - \psi(c)$. We are particularly interested in the case $c \geq \Phi(0)$ and here a reparameterisation is useful.

For $q \geq 0$ we will denote by $\mathbb{P}^{(q)}$ the measure under which X is a spectrally negative Lévy process with exponent

$$\psi^{(q)}(\lambda) = \psi(\lambda + \Phi(q)) - q,$$

which satisfies, for every $A \in \mathcal{F}_t$,

$$\mathbb{P}^{(q)}\{A \cap (X_t \in dx)\} = e^{-qt} e^{x\Phi(q)} \mathbb{P}\{A \cap (X_t \in dx)\}. \tag{9.2.10}$$

This measure has the following important property:

Lemma 11. *For every $x > 0$ and $q > 0$ the law of $(X_t, 0 \leq t < T_x)$ is the same under $\mathbb{P}^{(q)}$ as under $\mathbb{P}(\,\cdot\,|T_x < \boldsymbol{e}_q)$.*

Proof. Simply compute, for $y < x$ and $A \in \mathcal{F}_t$,

$$\begin{aligned}
&\mathbb{P}\{A \cap (X_t \in dy) \cap (t < T_x) | T_x < \boldsymbol{e}_q) \\
&\qquad = e^{-qt}\mathbb{P}\{A \cap (X_t \in dy) \cap (t < T_x)\}\mathbb{P}_y(T_x < \boldsymbol{e}_q)/\mathbb{P}(T_x < \boldsymbol{e}_q) \\
&\qquad = e^{-qt}\mathbb{P}\{A \cap (X_t \in dy) \cap (t < T_x)\}e^{y\Phi(q)} \\
&\qquad = \mathbb{P}^{(q)}\{A \cap (X_t \in dy) \cap (t < T_x)\},
\end{aligned}$$

where we have used (9.2.10). ■

Notice that $\mathbb{E}^{(q)}X_1 = \psi'(\Phi(q)) > 0$ when $q > 0$ or $q = 0$ and $\Phi(0) > 0$, and $\mathbb{P}^{(q)}$ agrees with $\mathbb{P}$ for $q = 0$ if $\Phi(0) = 0$. In the case $q = 0$ and $\Phi(0) > 0$ we will denote $\mathbb{P}^{(q)}$ by $\mathbb{P}^{\#}$, and call it the **associated** Lévy measure, with exponent

$$\psi^{\#}(\lambda) := \psi(\Phi(0) + \lambda).$$

Under $\mathbb{P}^{\#}$, X drifts to ∞, and is in fact a version of the original process conditioned to drift to ∞, in the sense that

$$\lim_{x\to\infty} \mathbb{P}(A|S_\infty > x) = \mathbb{P}^{\#}(A), \text{ for all } A \in \mathcal{F}_t, \text{ any } t > 0.$$

As such it constitutes a device which allows us to deduce results for spectrally negative Lévy process which drift to $-\infty$ from results for spectrally negative

Lévy process which drift to ∞, and sometimes vice versa. Note also that if $\Phi(0) > 0$, the $q = 0$ analogue of Lemma 11 is correct, viz for every $x > 0$ the law of $(X_t, 0 \le t < T_x)$ is the same under $\mathbb{P}^{\#}$ as under $\mathbb{P}(\,\cdot\,|T_x < \infty)$.

9.3 The Random Walk Case

The discrete analogue of a spectrally negative Lévy process is a upwards skip-free random walk. This is a random walk whose step-distribution is concentrated on the integers, and it is "discretely upwards continuous", in the sense that it has to visit $1, 2, \cdots, n-1$, before visiting $n \ge 1$. With $p_n = F(\{n\})$ it is clear that $E(e^{\lambda S_n}) = \pi(\lambda)^n$for $\lambda \ge 0$, where

$$\pi(\lambda) = E(e^{\lambda Y_1}) = \sum_{-\infty}^{1} p_n e^{n\lambda} < \infty.$$

Since the only possible value of H_1^+, the first strict inceasing ladder height, is 1, the spatial Wiener–Hopf factorisation (4.2.3) in Chapter 4 can be written as

$$1 - \pi(\lambda) = (1 - he^{\lambda})(1 - E(e^{-\lambda H_1^-})), \tag{9.3.1}$$

where $h = P(H_1^+ = 1)$.

As in Chapter 5, Section 5, let $D_1, D_2, \cdots$ denote the depths of the excursions below the maximum. Then for integers $y > x > 0$,

$$\begin{aligned}
&P_x(S \text{ hits } \{y\} \text{ before } \{\cdots, -2, -1, 0\}) \\
&\quad = P(D_1 < x, D_2 < x+1, \cdots D_{y-x} < y-1) \\
&\quad = \prod_1^{y-x} P(D_1 < x - 1 + r) = \frac{\prod_1^{y-1} P(D_1 < r)}{\prod_1^{x-1} P(D_1 < r)} \\
&\quad = \frac{\omega(x)}{\omega(y)}, \text{ where } \omega(x) = \frac{1}{\prod_1^{x-1} P(D_1 < r)}.
\end{aligned}$$

This solves the two-sided exit problem, and should be compared to the upcoming (9.4.2) and (9.4.5). ω is the discrete version of the scale function, and in this situation we can see analogues of several results which figure in the following sections.

- When $S \overset{a.s.}{\to} \infty$ we can write

$$\omega(x) = \frac{\prod_x^{\infty} P(D_1 < r)}{\prod_1^{\infty} P(D_1 < r)} = \frac{P(I_\infty \ge -x)}{P(I_\infty = 0)}, \tag{9.3.2}$$

and using (9.3.1) (note that $h = 1$) we can check that

$$\int_0^\infty e^{-\lambda x}\omega(x)dx = \frac{e^\lambda - 1}{\lambda(\pi(\lambda) - 1)}. \tag{9.3.3}$$

(Compare the upcoming (9.4.3) and (9.4.1).)

- Let D_1^* denote the height of the first excursion above the minimum: then

$$\begin{aligned} P(D_1^* \geq y) &= P_0(S \text{ hits } \{y\} \text{ before } \{\cdots, -2, -1, 0, \}) \\ &= p_1\ P_1(S \text{ hits } \{y\} \text{ before } \{\cdots, -2, -1, 0, \}) \\ &= \frac{p_1\omega(1)}{\omega(y)}, \end{aligned}$$

so that we have the alternate expression

$$\omega(y) = \frac{c}{P(D_1^* \geq y)};$$

compare Corollary 14, part (ii).

9.4 The Scale Function

In what follows W will denote the scale function, which we will see is the unique absolutely continuous increasing function with Laplace transform

$$\int_0^\infty e^{-\lambda x}W(x)dx = \frac{1}{\psi(\lambda)},\ \lambda > \Phi(0). \tag{9.4.1}$$

The following result is contained in Takács [96]: the proof there relies on random-walk approximation. The first Lévy process proof is due to Emery in [45], where complicated complex variable techniques are used. Later proofs are in Rogers, [86] and [87], and Bertoin [12], Section VII.2. Define for $a \geq 0$ the passage times

$$T_a = \inf(t \geq 0 : X_t > a),\ T_a^* = \inf(t \geq 0 : -X_t > a).$$

Theorem 30. *For every $0 < x < a$, the probability that X, starting from x, makes its first exit from $[0, a]$ at a is*

$$\mathbb{P}_x(T_a < T_0^*) = \frac{W(x)}{W(a)}. \tag{9.4.2}$$

Example 3. *If X is a standard spectrally negative stable process then $\psi(\lambda) = \lambda^\alpha$, where $1 < \alpha \leq 2$, and $W(x) = \frac{x^{\alpha-1}}{\Gamma(\alpha)}$.*

Proof. The following observation is used in [86]; see also Kyprianou and Palmowski [70] and Kyprianou [69], Chapter 8. Suppose first that $\mathbb{E}X_1 > 0$; we will show that the function defined by

$$W(x) = \frac{\mathbb{P}(I_\infty \geq -x)}{\psi'(\Phi(0))} = \frac{\mathbb{P}(I_\infty \geq -x)}{\psi'(0)} \tag{9.4.3}$$

satisfies both (9.4.1) and (9.4.2). An integration by parts and (9.2.7) give

$$\int_0^\infty e^{-\lambda x} W(x)dx = \frac{1}{\lambda\psi'(\Phi(0))} \int_0^\infty e^{-\lambda x} \mathbb{P}(-I_\infty \in dx) = \frac{1}{\psi(\lambda).}$$

Also

$$\mathbb{P}(T_{a-x} < T_x^*) = \mathbb{P}(I(T_{a-x}) \geq -x).$$

However, by the strong Markov property applied at time T_{a-x}, which is a.s. finite because $\mathbb{E}X_1 > 0$,

$$\mathbb{P}(I_\infty \geq -x) = \mathbb{P}(I(T_{a-x}) \geq -x)\mathbb{P}(I_\infty \geq -a),$$

so we see that (9.4.2) holds. Next, if X drifts to $-\infty$, we claim that

$$W(x) = e^{\Phi(0)x} W^\#(x), \tag{9.4.4}$$

where $W^\#$ denotes W evaluated under the associated measure $\mathbb{P}^\#$ introduced at the end of the previous section. We have

$$\begin{aligned} \int_0^\infty e^{-\lambda x} W(x)dx &= \int_0^\infty e^{-(\lambda - \Phi(0))x} W^\#(x)dx \\ &= \frac{1}{\psi^\#(\lambda - \Phi(0))} = \frac{1}{\psi(\lambda)}, \end{aligned}$$

and, by the final remark in the previous section

$$\begin{aligned} \mathbb{P}(T_{a-x} < T_x^*) &= e^{-(a-x)\Phi(0)} \mathbb{P}(T_{a-x} < T_x^* | T_{a-x} < \infty) \\ &= e^{-(a-x)\Phi(0)} \mathbb{P}^\#(T_{a-x} < T_x^*) = \frac{e^{x\Phi(0)} W^\#(x)}{e^{a\Phi(0)} W^\#(a)} = \frac{W(x)}{W(a)}. \end{aligned}$$

When X oscillates, some kind of limiting argument is necessary, and the most satisfactory seems to be the following, which is taken from [45]. Let $\tilde{\mathbb{P}}^{(\varepsilon)}$ be the measure corresponding to the process $X_t + \varepsilon t$, where $\varepsilon > 0$, and note that, in the obvious notation, $\tilde{\psi}^{(\varepsilon)}(\lambda) \to \psi(\lambda)$ as $\varepsilon \downarrow 0$. So, using the continuity theorem for Laplace transforms, we deduce from (9.4.1) that $W(x) = \lim_{\varepsilon \downarrow 0} \tilde{W}^{(\varepsilon)}(x)$ exists. To show that (9.4.2) holds with this W, note that

$$\mathbb{P}(T_{a-x} < T_x^*) \leq \tilde{\mathbb{P}}^{(\varepsilon)}(T_{a-x} < T_x^*) = \frac{\tilde{W}^{(\varepsilon)}(x)}{\tilde{W}^{(\varepsilon)}(a)}.$$

On the other hand, for fixed $0 < t < x/\varepsilon$,

$$\mathbb{P}(T_x^* < t, T_x^* < T_{a-x}) \le \tilde{\mathbb{P}}^{(\varepsilon)}(T_{x-\varepsilon t}^* < T_{a-x}) = 1 - \frac{\tilde{W}^{(\varepsilon)}(x)}{\tilde{W}^{(\varepsilon)}(a-\varepsilon t)},$$

and the conclusion follows by letting $\varepsilon \downarrow 0$ and then $t \to \infty$. ■

In [12], Section VII.2, an excursion argument is used to show that if X drifts to ∞ we have the representation

$$W(x) = c \exp\{-\int_x^\infty \overline{n}(t < h(\varepsilon) < \infty)dt\}, \tag{9.4.5}$$

where $\overline{n}$ denotes the characteristic measure of the Poisson point process of the excursions of $S - X$ away from 0 and $h(\varepsilon)$ denotes the height of a typical excursion ε. It is also claimed that (9.4.5) also holds in the oscillatory case. But actually $\int_x^\infty \overline{n}(t < h(\varepsilon) < \infty)dt = \infty$ when X oscillates; for example for Brownian motion we have $\overline{n}(t < h(\varepsilon) < \infty) = 1/t$. (See e.g. (ii) in Corollary 14 below.)

The proof in Rogers [86] claims that in the oscillatory case

$$W(x) = \lim_{y \to \infty} \mathbb{P}(I_{T_y} \ge -x | T_y < \infty);$$

of course, in this case the conditioning is redundant, and $I_\infty = -\infty$ a.s., so the right-hand side is actually zero.

By comparing Laplace transforms, we also see that

$$W(x) = \begin{cases} cU^*(x) & \text{if } \Phi(0) = 0, \\ ce^{\Phi(0)x}U^{\#*}(x) & \text{if } \Phi(0) > 0, \end{cases}, \tag{9.4.6}$$

where U^*, $U^{\#*}$ are the potential functions for the ladder process H^* under $\mathbb{P}$ and $\mathbb{P}^{\#}$.

Since H is a pure drift, with killing if $\Phi(0) > 0$, we have $U(dx) = e^{-\Phi(0)x}dx$, so the équation amicale inversée (5.3.4) takes the simple form

$$\begin{aligned} \overline{\mu^*}(x) &= \int_0^\infty e^{-\Phi(0)y}\overline{\Pi}^*(x+y)dy \\ &= \int_x^\infty \overline{\Pi}^*(y)dy \quad \text{if } X \text{ does not } \to -\infty. \end{aligned} \tag{9.4.7}$$

Another couple of useful facts are contained in the following:

Corollary 14. *(i) For each $x > 0$ the process*

$$W(X_t)\mathbf{1}_{\{T_0^* > t\}}$$

is a $\mathbb{P}_x$-martingale.

(ii) If $\underline{n}$ denotes the characteristic measure of the Poisson point process of the excursions of $X - I$ away from 0 we have, for some $c > 0$ and all $x > 0$,

$$\underline{n}(x < h(\varepsilon) < \infty) = \frac{c}{W(x)}.$$

Proof. (i) When X doesn't drift to $-\infty$ the observation (9.4.6) shows that this is a special case of Lemma 10, Chapter 8, and when X does drift to $-\infty$ we can verify it by using the device of the associated process.

(ii) Once we recognise that for fixed $y > 0$

$$\underline{n}(\cdot|y < h(\varepsilon) < \infty) = \frac{\underline{n}(\cdot \cap (y < h(\varepsilon) < \infty))}{\underline{n}(y < h(\varepsilon) < \infty)}$$

is a probability measure which, by the Markov property, coincides with $\mathbb{P}_y$, this follows from Theorem 30. ■

Although Theorem 30 apparently solves completely the 2-sided exit problem, it is not necessarily easy to exploit it.

Example 4. *Exit from a symmetric interval. It would seem that it should be easy to ascertain the limiting probability that a spectrally negative Lévy process exits a symmetric interval at the top. Specifically the question is when does* $\pi(x) \to \rho \in [0,1]$ *as* $x \to \infty$*, where by Theorem 30*

$$\pi(x) := \mathbb{P}_0(T_x < T_x^*) = \frac{W(x)}{W(2x)}.$$

Clearly $\pi(x) \to 1$, *(respectively* 0*), if* X *drifts to* ∞ *(respectively* $-\infty$*), so assume* X *oscillates, i.e.* $\mathbb{E}X_1 = 0$. *Then* W *is a multiple of the potential function* U^* *of* H^**, and therefore is subadditive. Thus* $W(2x) \leq 2W(x)$*, so always* $\pi(x) \geq 1/2$. *If* $W \in RV(\kappa)$ *at* ∞ *then* $\pi(x) \to 2^{-\kappa}$ *and from the defining relation (9.4.1) we have*

$$\int_0^\infty e^{-\lambda x} W(dx) = \frac{\lambda}{\psi(\lambda)},$$

so this happens if and only if $\psi \in RV(1+\kappa)$ *at 0, which is possible for any* $0 \leq \kappa \leq 1$. *On the other hand, if we could deduce from*

$$\pi(x) = \frac{W(x)}{W(2x)} \to \rho = \frac{1}{2^\kappa} \tag{9.4.8}$$

that $W \in RV(\kappa)$*, we would be able to reverse the argument, thus getting a necessary and sufficient condition for (9.4.8) to hold. However, in general we need to have* $W(x)/W(cx) \to c^{-\kappa}$ *for two values of* c *which are such that the ratio of their logarithms is irrational (see [21]) to draw this conclusion, and I know no way of establishing this. So we do NOT KNOW if* $\pi(x) \to \rho \in [1/2, 1)$ *and* ψ *not regularly varying can occur. When* $\rho = 1$ *we can argue that for any* $1 < c \leq 2$

$$1 \geq \frac{W(x)}{W(cx)} \geq \frac{W(x)}{W(2x)},$$

so $W(x)/W(2x) \to 1$ *if and only if* $W(x)$ *is slowly varying as* $x \to \infty$*, or equivalently* $\psi \in RV(1)$*, but this is clearly an easier case.*

We can write $\pi(x) = \mathbb{P}(X(\gamma_x) > 0)$, where γ_x denotes the exit time from $[-x, x]$, so there might be some relation between the convergence of $\pi(x)$ and the convergence of $\mathbb{P}(X_t > 0)$. However we know that this last is equivalent to Spitzer's condition, and this in turn is equivalent to the regular variation of Φ. Since this Φ is the inverse of ψ, we can conclude (see Proposition 6, p. 192 of [12]) that for $1/2 \le \rho < 1$

$$\begin{aligned}\psi \ &\in \ RV(1/\rho) \Longleftrightarrow \mathbb{P}(X_t > 0) \to \rho \text{ as } t \to \infty \\ &\Longrightarrow \pi(x) \to 2^{1-\frac{1}{\rho}} \text{ as } x \to \infty,\end{aligned}$$

and for $\rho = 1$,

$$\begin{aligned}\psi \ &\in \ RV(1) \Longleftrightarrow \mathbb{P}(X_t > 0) \to 1 \text{ as } t \to \infty \\ &\Longleftrightarrow \pi(x) \to 1 \text{ as } x \to \infty\end{aligned}$$

It is also possible to express the condition $\psi \in RV(1/\rho)$ in terms of the Lévy measure of X; for example, when $\rho = 1$, it is equivalent to $\int_x^\infty \overline{\Pi^*}(y)dy$ being slowly varying as $x \to \infty$.

9.5 Further Developments

Another interesting object connected to the 2-sided exit problem is the overshoot, and the results in the previous section give no information about this, other than the value of its mean. It seems that to obtain more information, it is necessary to study also the exit time $\sigma_a = T_a \wedge T_0^*$.

In Bertoin [15] the author exploited the fact that the q-scale function $W^{(q)}$, which informally is the scale function of the process got by killing X at an independent Exp(q) time, determines also the distribution of this exit time. Specifically $W^{(q)}$ denotes the unique absolutely continuous increasing function with Laplace transform

$$\int_0^\infty e^{-\lambda x} W^{(q)}(x)dx = \frac{1}{\psi(\lambda) - q}, \ \lambda > \Phi(q), q \ge 0, \tag{9.5.1}$$

and for convenience we set $W^{(q)}(x) = 0$ for $x \in (-\infty, 0)$. We also need the function defined by $Z^{(q)}(x) = 1$ for $x \le 0$ and

$$Z^{(q)}(x) = 1 + q\overline{W}^{(q)}(x) \text{ for } x > 0, \text{ where } \overline{W}^{(q)}(x) = \int_0^x W^{(q)}(y)dy. \tag{9.5.2}$$

Extending previous results due to Takács [96], Emery [45], Suprun [95], Koryluk et al [67], and Rogers [86], Bertoin [15] gave the full solution to the 2-sided exit problem in the following form:

Theorem 31. *For $0 \le x \le a$ and $q \ge 0$ we have*

$$\mathbb{E}_x(e^{-qT_a}; T_a < T_0^*) = \frac{W^{(q)}(x)}{W^{(q)}(a)}, \tag{9.5.3}$$

and

$$\mathbb{E}_x(e^{-qT_0^*}; T_0^* < T_a) = Z^{(q)}(x) - \frac{W^{(q)}(x)Z^{(q)}(a)}{W^{(q)}(a)}. \tag{9.5.4}$$

Furthermore let $U^{(q)}$ denote the resolvent measure of X killed at time σ_a. Then $U^{(q)}$ has a density which is given by

$$u^{(q)}(x,y) = \frac{W^{(q)}(x)}{W^{(q)}(a)} W^{(q)}(a-y) - W^{(q)}(x-y), \; x,y \in [0,a). \tag{9.5.5}$$

Remark 5. *(i) From (9.5.5) we can immediately write down the joint distribution of the exit time and overshoot, since the compensation formula gives, for $x, y \in (0,a)$ and $z \le 0$,*

$$\mathbb{E}_x(e^{-q\sigma_a}; X(\sigma_a-) \in dy, X(\sigma_a) \in dz) = u^{(q)}(x,y)dy\Pi(dz-y).$$

Note that this holds even for $q = 0$.

(ii) It seems obvious that by letting $a \to \infty$ we should be able to get the distribution of the downward passage time T_0^* under $\mathbb{P}_x, x > 0$. As we will see below, it is in fact true that

$$\mathbb{E}_x(e^{-qT_0^*}; T_0^* < \infty) = Z^{(q)}(x) - \frac{qW^{(q)}(x)}{\Phi(q)}. \tag{9.5.6}$$

However to deduce this directly from (9.5.4) we need to know that $\frac{Z^{(q)}(a)}{W^{(q)}(a)} \to \frac{q}{\Phi(q)}$ as $a \to \infty$, which requires some work.

Proof. Take $q > 0$. Using Lemma 11, we see that

$$\begin{aligned}\mathbb{E}_x(e^{-qT_a}; T_a < T_0^*) &= \mathbb{P}(T_{a-x} < \boldsymbol{e}_q; T_{a-x} < T_x^*) \\ &= e^{-(a-x)\Phi(q)}\mathbb{P}(I(T_{a-x}) \ge -x | T_{a-x} < \boldsymbol{e}_q) \\ &= e^{-(a-x)\Phi(q)}\mathbb{P}^{(q)}(I(T_{a-x}) \ge -x).\end{aligned}$$

However, X drifts to ∞ under $\mathbb{P}^{(q)}$, so if we define

$$W^{(q)}(x) = c(q)e^{x\Phi(q)}\mathbb{P}^{(q)}(I_\infty \ge -x) \tag{9.5.7}$$

we see from Theorem 30 that (9.5.3) holds. Moreover taking $\lambda > \Phi(q)$ and writing $\tilde{\lambda} = \lambda - \Phi(q)$, it follows from (9.2.7) that

$$\int_0^\infty e^{-\lambda x} W^{(q)}(x)dx = c(q) \int_0^\infty e^{-\tilde{\lambda} x} \mathbb{P}^{(q)}(I_\infty \geq -x)dx$$
$$= \frac{c(q)}{\tilde{\lambda}} \mathbb{E}^{(q)}(e^{\tilde{\lambda} I_\infty}) = \frac{c(q)\psi'(\Phi(q))}{\psi^{(q)}(\tilde{\lambda})}$$
$$= \frac{c(q)\psi'(\Phi(q))}{(\psi(\lambda) - q)}.$$

So if we choose $c(q) = 1/\psi'(\Phi(q))$ we have (9.5.1) for $q > 0$. Still keeping $q > 0$ we can use (9.5.3) in (9.2.6) to deduce that

$$\mathbb{P}(-I_{\mathbf{e}(q)} \in dx) = \frac{q}{\Phi(q)} W^{(q)}(dx) - qW^{(q)}(x)dx.$$

(Note that (9.5.6) follows quickly from this.) Also, by the Wiener–Hopf factorisation, $I_{\mathbf{e}(q)}$ and $X_{\mathbf{e}(q)} - I_{\mathbf{e}(q)}$ are independent, and the latter has the distribution of $S_{\mathbf{e}(q)}$, which is $\mathrm{Exp}(\Phi(q))$. This allows us to compute that, for $x, y > 0$,

$$\mathbb{P}_x(X_{\mathbf{e}(q)} \in dy, I_{\mathbf{e}(q)} > 0) = q\left(e^{-\Phi(q)y} W^{(q)}(x) - W^{(q)}(x-y)\right) dy, \quad (9.5.8)$$

where we recall that $W^{(q)}(x) = 0$ for $x < 0$. Then applying the strong Markov property at time σ_a gives

$$qu^{(q)}(x,y) = \mathbb{P}_x(X_{\mathbf{e}(q)} \in dy, \mathbf{e}(q) < \sigma_a) = \mathbb{P}_x(X_{\mathbf{e}(q)} \in dy, I_{\mathbf{e}(q)} > 0)$$
$$-\mathbb{P}_x(X_{\sigma_a} = a, \sigma_a < \mathbf{e}(q))\mathbb{P}_a(X_{\mathbf{e}(q)} \in dy, I_{\mathbf{e}(q)} > 0),$$

and (9.5.5) follows from (9.5.3) and (9.5.8). Integrating (9.5.5) over $(0, a)$ gives $\mathbb{P}_x(\mathbf{e}(q) < \sigma_a)$, and subtracting (9.5.3) from $1 - \mathbb{P}_x(\mathbf{e}(q) < \sigma_a)$ gives (9.5.4). We can then let $q \downarrow 0$ to see that (9.5.4) and (9.5.5) also hold for $q = 0$. ■

A simple, but crucial remark, is that

$$\frac{1}{\psi(\lambda) - q} = \sum_{k=0}^{\infty} q^k \psi(\lambda)^{-k-1}, \ \lambda > \Phi(q),$$

and by Laplace inversion we have the following representation for $W^{(q)}$:

$$W^{(q)}(x) = \sum_{k=0}^{\infty} q^k W^{*(k+1)}(x), \quad (9.5.9)$$

where $W^{*(n)}$ denotes the nth convolution power of the scale function W. (Note that the bound

$$W^{*(k+1)}(x) \leq \frac{x^k W(x)^{k+1}}{k!} \quad (9.5.10)$$

justifies this argument.)

In the stable case we can check that

$$W^{*(n)}(x) = \frac{x^{n\alpha-1}}{\Gamma(n\alpha)}, \text{ so that}$$

$$W^{(q)}(x) = \sum_{k=0}^{\infty} \frac{q^k x^{(k+1)\alpha-1}}{\Gamma((k+1)\,\alpha)} = \alpha x^{\alpha-1} E'_\alpha(qx^\alpha), \tag{9.5.11}$$

where E'_α is the derivative of the Mittag-Leffler function of parameter α,

$$E_\alpha(y) = \sum_{k=0}^{\infty} \frac{y^k}{\Gamma(k\alpha+1)}, \; y \in \mathbb{R}.$$

In particular, for $\alpha = 2$, $X/\sqrt{2}$ is a standard Brownian motion,

$$E'_2(x) = \frac{\sinh\sqrt{x}}{2\sqrt{x}}, \text{ and } W^{(q)}(x) = \frac{\sinh x\sqrt{q}}{\sqrt{q}}. \tag{9.5.12}$$

As well as giving the above derivation (an earlier proof, in [95], was heavily analytic and published in Russian), Bertoin [15] showed how these results can be exploited to yield important information about the exit time σ_a, whose distribution is specified by

$$\mathbb{E}_x\{\exp(-q\sigma_a)\} = 1 + q\left\{\overline{W}^{(q)}(x) - \frac{W^{(q)}(x)\overline{W}^{(q)}(a)}{W^{(q)}(a)}\right\}. \tag{9.5.13}$$

In fact he showed that, modulo some minor regularity conditions, in **all cases the tail has an exact exponential decay**.

The key to this is to study $W^{(q)}(x)$ as a function of q on the negative half-line; in the special case of Brownian motion, one easily verifies that for each $x > 0$ we can extend $W^{(q)}(x)$ analytically to the negative q-axis, (in fact $W^{(-q)}(x) = (\sin\sqrt{q}x)/\sqrt{q}$ for $q > 0$), that $W^{(-q)}(x)$ has a simple zero at $q = \rho(x) = (\pi/x)^2$ and is positive on $[0, \rho(x))$. One can then conclude from (9.5.10) that, with $\rho = \rho(a)$,

$$1 - \mathbb{E}_x\{\exp(-(q-\rho)\sigma_a)\} \sim \frac{c}{q} \text{ as } q \downarrow 0.$$

This statement is compatible with the desired conclusion that

$$\lim_{t\to\infty} e^{\rho t}\mathbb{P}_x(\sigma_a > t) \text{ is finite,} \tag{9.5.14}$$

but it doesn't seem possible to establish this implication by means of a Tauberian theorem. Indeed I don't think (9.5.14) was known even in the Brownian case. In Bertoin [13] a weaker version of (9.5.14) was obtained in the stable case; here an interesting feature is the way that ρ depends on α, taking its minimum value when $\alpha \simeq 1.26$.

However, returning to the problem in [15], Bertoin showed that (9.5.14) is in fact true in general. Interestingly, this was accomplished not by analytic arguments, but by showing that the process killed at time σ_a is a ρ-positive recurrent strong Markov process.

Theorem 32. *Assume the absolute continuity condition*

$$\mathbb{P}_0(X_t \in dx) << dx \text{ for any } t > 0,$$

and write

$$P^t(x, A) = \mathbb{P}_x(X_t \in A, \sigma_a > t).$$

Define

$$\rho = \inf\{q \geq 0 : W^{(-q)}(a) = 0\}.$$

Then ρ is finite and positive and $W^{(-q)}(x) > 0$ for any $q < \rho$ and $x \in (0, a)$. Furthermore

(i) ρ is a simple root of the entire function $W^{(-q)}(a)$;
(ii) P^t is ρ-positive recurrent;
(iii) the function $W^{(-\rho)}(\cdot)$ is positive on $(0, a)$ and ρ-invariant for P^t,

$$P^t W^{(-\rho)}(x) = e^{-\rho t} W^{(-\rho)}(x);$$

(iv) the measure $\mu(dx) = W^{(-\rho)}(a - x)dx$ on $(0, a)$ is ρ-invariant for P^t,

$$\mu P^t(dx) = e^{-\rho t}\mu(dx);$$

(v) there is a constant $c > 0$ such that for any $x \in (0, a)$

$$\lim_{t\to\infty} e^{\rho t} P^t(x, \cdot) = \frac{1}{c} W^{(-\rho)}(x)\mu(\cdot)$$

in the sense of weak convergence.

Suppose we define

$$D_t = e^{\rho t}\mathbf{1}_{\{\sigma_a > t\}}\frac{W^{(-\rho)}(X_t)}{W^{(-\rho)}(x)},\ 0 < x < a.$$

Then using (iii) above we can check that

$$\begin{aligned}
\mathbb{E}_x(D_{t+s}|\mathcal{F}_t) &= \frac{e^{\rho(t+s)}}{W^{(-\rho)}(x)}\mathbb{E}_x(\mathbf{1}_{\{\sigma_a > t+s\}}W^{(-\rho)}(X_{t+s})|\mathcal{F}_t) \\
&= \frac{e^{\rho(t+s)}}{W^{(-\rho)}(x)}\mathbf{1}_{\{\sigma_a > t\}}\mathbb{E}_{X_t}(\mathbf{1}_{\{\sigma_a > s\}}W^{(-\rho)}(X_s)) \\
&= \frac{e^{\rho(t+s)}}{W^{(-\rho)}(x)}\mathbf{1}_{\{\sigma_a > t\}}W^{(-\rho)}(X_t)e^{-\rho s} = D_t,
\end{aligned}$$

so D is a $\mathbb{P}$-martingale. Just as $W(X_t)\mathbf{1}_{\{T_0^* > t\}}$ can be used to construct a version of X conditioned to stay positive, so D can be used to construct a version of X conditioned to remain within the interval $(0, a)$. This programme was carried out in Lambert [72], where some further properties of the conditioned process were also derived.

9.6 Exit Problems for the Reflected Process

Recently, because of potential applications in mathematical finance, there has been considerable interest in the possibility of solving exit problems involving the reflected processes defined by

$$Y_t = X_t - \underline{X}_t,\ Y_t^* = \overline{X}_t - X_t,\ t \geq 0.$$

In Avram, Kyprianou and Pistorius [8] and Pistorius [82] some new results about the times at which Y and Y^* exit from finite intervals have been deduced from Theorem 31. The proofs of these results in the cited papers involve a combination of excursion theory, Itô calculus, and martingale techniques, and in [35] I showed that these results can be established by direct excursion-theory calculations. (See also [81] and [78] for different approaches.) My arguments are also based on Theorem 31, but the other ingredient is the representation for the characteristic measure $\underline{n}$ of the excursions of Y away from zero given in Chapter 8. Here I will explain the basis of my calculations, without going into all the details.

Let X be **any** Lévy process with the property that 0 is regular for $\{0\}$ for Y, and introduce the excursion measure $\underline{n}$ and the harmonic function h as in Section 2 of Chapter 8. In the following result ζ denotes the lifetime of an excursion and $\mathbb{Q}_x$ denotes the law of X killed on entering $(-\infty, 0)$.

Proposition 17. *Let $A \in \mathcal{F}_t, t > 0$, be such that $\underline{n}(A^o) = 0$, where A^o is the boundary of A with respect to the J-topology on D. Then for some constant k (which depends only on the normalization of the local time at zero of Y),*

$$\underline{n}(A, t < \varsigma) = k \lim_{x \downarrow 0} \frac{\mathbb{Q}_x(A)}{h(x)}. \tag{9.6.1}$$

Proof. According to Corollary 12, Section 4 of Chapter 8, for any $A \in \mathcal{F}_t$ we have

$$\underline{n}(A, t < \varsigma) = k\mathbb{E}^{\uparrow}(h(X_t)^{-1}; A), \tag{9.6.2}$$

where $\mathbb{P}^{\uparrow}$ is the weak limit in the Skorohod topology as $x \downarrow 0$ of the measures $\mathbb{P}_x^{\uparrow}$ which correspond to "conditioning X to stay positive", and are defined by

$$\mathbb{P}_x^{\uparrow}(X_t \in dy) = \frac{h(y)}{h(x)} \mathbb{Q}_x(X_t \in dy),\ x > 0, y > 0.$$

Combining these results and using the assumption on A gives (9.6.1). ■

(Since we will only be concerned with ratios of $\underline{n}$ measures in the following we will assume that $k = 1$.)

The relevance of this is that the results in Theorem 31 are in fact results about $\mathbb{Q}_x$, and moreover if now X is a spectrally negative Lévy process which does not drift to $-\infty$, then $h(x) = U^*(x) = W(x)$, which means it may be possible to compute the $\underline{n}$-measures of certain sets. Put $\eta(\varepsilon) := \sup_{t<\zeta} \varepsilon(t)$ and $T_a(\varepsilon) = \inf\{t : \varepsilon(t) > a\}$ for the height and the first passage time of a generic excursion ε whose lifetime is denoted by $\varsigma(\varepsilon)$, and with $\boldsymbol{e}_q$ denoting an independent Exp(q) random variable set $A = \{T_a(\varepsilon) \wedge \boldsymbol{e}_q < \varsigma(\varepsilon)\} = A_1 \cup A_2$, where

$$A_1 = \{\varepsilon : \eta(\varepsilon) > a, T_a(\varepsilon) < \varsigma(\varepsilon) \wedge \boldsymbol{e}_q\},$$

and

$$A_2 = \{\varepsilon : \eta(\varepsilon) \le a, \boldsymbol{e}_q < \varsigma(\varepsilon)\}.$$

Noting that $\underline{n}(\eta(\varepsilon) > x) = c/W(x)$ is continuous, we can apply (9.6.1) to see that

$$\begin{aligned}\underline{n}(\eta(\varepsilon) > a, T_a(\varepsilon) \in dt) &= \lim_{x\downarrow 0} \frac{\mathbb{Q}_x\{T_a \in dt\}}{W(x)} \\ &= \lim_{x\downarrow 0} \frac{\mathbb{P}_x\{T_a < T_0^*, T_a \in dt\}}{W(x)},\end{aligned}$$

and

$$\underline{n}\{T_a(\varepsilon) > t) = \lim_{x\downarrow 0} \frac{\mathbb{Q}_x\{T_a > t\}}{W(x)} = \lim_{x\downarrow 0} \frac{\mathbb{P}_x\{\sigma_a > t\}}{W(x)}.$$

Thus

$$\begin{aligned}\underline{n}(A) &= \underline{n}(A_1) + \underline{n}(A_2) \\ &= \lim_{x\downarrow 0} \frac{1}{W(x)} \left(\mathbb{E}_x\{e^{-qT_a}; T_a < T_0^*\} + \mathbb{P}_x\{\boldsymbol{e}_q < \sigma_a\}\right) \\ &= \lim_{x\downarrow 0} \frac{1}{W(x)} \left(1 - \mathbb{E}_x\{e^{-qT_0^*}; T_0^* < T_a\}\right).\end{aligned}$$

Combining this with (9.5.4) gives

$$\underline{n}(A) = \lim_{x\downarrow 0} \frac{1 - Z^{(q)}(x)}{W(x)} + \frac{Z^{(q)}(a)}{W^{(q)}(a)} \lim_{x\downarrow 0} \frac{W^{(q)}(x)}{W(x)} = \frac{Z^{(q)}(a)}{W^{(q)}(a)}. \tag{9.6.3}$$

In a similar way it follows from (9.5.5) that

$$\begin{aligned}\underline{n}\{\boldsymbol{e}_q < \zeta, \varepsilon(\boldsymbol{e}_q) \in dy, \overline{\varepsilon}(\boldsymbol{e}_q) \le a\} &= \lim_{x\downarrow 0} \frac{\mathbb{P}_x\{\boldsymbol{e}_q < \sigma_a, X(\boldsymbol{e}_q) \in dy\}}{W(x)} \\ = \lim_{x\downarrow 0} \frac{u^{(q)}(x,y)dy}{W(x)} &= \lim_{x\downarrow 0} \frac{W^{(q)}(a-y)W^{(q)}(x)dy}{W(x)W^{(q)}(a)} \\ = \frac{W^{(q)}(a-y)dy}{W^{(q)}(a)}.\end{aligned}$$

Note that for subsets B of A, $\underline{n}(B)/\underline{n}(A)$ is a probability measure, which excursion theory tells us coincides with

$$\mathbb{P}(Y(\underline{L}^{-1}(\hat{t}) + \cdot) \in B),$$

where $\hat{t} = \inf(s : \varepsilon_s(\cdot) \in A)$ is the local time of the first excursion which either exits $[0, a]$ or spans $\boldsymbol{e}_q$. In particular, if $T_a = \inf\{t : Y_t > a\}$ for $y \in (0, a)$ we have

$$\begin{aligned}
\mathbb{P}(T_a > \ \boldsymbol{e}_q, Y(\boldsymbol{e}_q) \in dy) &= \mathbb{P}(\overline{Y}(\boldsymbol{e}_q) \le a, Y(\ \boldsymbol{e}_q) \in dy) \\
&= \mathbb{P}(\overline{Y}(\boldsymbol{e}_q) \le a, Y(\boldsymbol{e}_q) \in dy) \\
&= \underline{n}\{\ \boldsymbol{e}_q < \zeta, \varepsilon(\boldsymbol{e}_q) \in dy, \overline{\varepsilon}(\boldsymbol{e}_q) \le a\}/\underline{n}(A) \\
&= \frac{W^{(q)}(a-y)dy}{W^{(q)}(a)} \cdot \frac{W^{(q)}(a)}{Z^{(q)}(a)} = \frac{W^{(q)}(a-y)dy}{Z^{(q)}(a)}.
\end{aligned}$$

This leads to the first part of the following result, which gives the q-resolvent measures $R^{(q)}(x, A)$ and $R^{*(q)}(x, A)$ of Y and Y^* killed on exiting the interval $[0, a]$.

Theorem 33. *(Pistorius) (i) The measure $R^{(q)}(x, A)$ is absolutely continuous with respect to Lebesgue measure and a version of its density is*

$$r^{(q)}(x, y) = \frac{Z^{(q)}(x)}{Z^{(q)}(a)} W^{(q)}(a-y) - W^{(q)}(x-y), \ \ x, y \in [0, a). \tag{9.6.4}$$

(ii) For $0 \le x \le a$ we have

$$R^{*(q)}(x, dy) = r^{*(q)}(x, 0)\delta_0(dy) + r^{*(q)}(x, y)dy,$$

where

$$r^{*(q)}(x, 0) = \frac{W^{(q)}(a-x)W^{(q)}(0)}{W_+^{(q)\prime}(a)}, \tag{9.6.5}$$

$$r^{*(q)}(x, y) = \frac{W^{(q)}(a-x)W_+^{(q)\prime}(y)}{W_+^{(q)\prime}(a)} - W^{(q)}(y-x), \tag{9.6.6}$$

$W_+^{(q)\prime}(y)$ denotes the right-hand derivative with respect to y of $W^{(q)}(y)$, and δ_0 denotes a unit mass at 0.

Proof. (i) This follows from the obvious decomposition

$$r^{(q)}(x, y) = u^{(q)}(x, y) + \mathbb{E}_x\{e^{-q\hat{T}_0}; T_0^* < T_a\} r^{(q)}(0, y),$$

and the previous calculation.

(ii) This follows a similar pattern to (i), and I will just explain how $W_+^{(q)'}$ enters the picture. First note that now we have $h(x) = U(x) = x$, and so the analogue of (9.6.3) is given by

$$\begin{aligned}\underline{n}^*(A) &= \lim_{x\downarrow 0}\frac{1}{x}\left(\mathbb{E}_{a-x}\{e^{-qT_0^*}; T_0^* < T_a\} + \mathbb{P}_{a-x}\{\,\boldsymbol{e}_q < \sigma_a\}\right) \\ &= \lim_{x\downarrow 0}\frac{1}{x}\left(1 - \mathbb{E}_{a-x}\{e^{-qT_a}; T_a < T_0^*\}\right) \\ &= \lim_{x\downarrow 0}\frac{1}{x}\left(\frac{W^{(q)}(a) - W^{(q)}(a-x)}{W^{(q)}(a)}\right) = \frac{W_+^{(q)'}(a)}{W^{(q)}(a)}.\end{aligned}$$

■

Remark 6. *We can deduce the joint distribution of the exit time and overshoot, just as we did for X.*

9.7 Addendum

There is one other special case where a similar idea works. The point is that some explicit results are known about the 2-sided exit problem in the case that X is a strictly stable process. In fact if we write σ for σ_1 and X is stable with parameter $0 < \alpha < 2, \alpha \neq 1$, and positivity parameter $\rho \in (1-1/\alpha, 1/\alpha)$, (so that we don't have a spectrally one-sided case) Rogozin [89] contains the following result. For $x \in (0,1)$, $y \in (1,\infty)$

$$\mathbb{P}_x(X_\sigma \in dy) = \frac{\sin\alpha\rho\pi}{\pi}\frac{(1-x)^{\alpha\rho}x^{\alpha(1-\rho)}dy}{(y-x)(y-1)^{\alpha\rho}y^{\alpha(1-\rho)}}. \tag{9.7.1}$$

(Note that we can get the corresponding result for downwards exit by considering $-X$, and for $\sigma_a, a \neq 1$ by scaling.) Since the downgoing ladder height process is a stable subordinator of index $\alpha(1-\rho)$, we can take $h(x) = x^{\alpha(1-\rho)}$, and rewrite (9.7.1) as

$$\mathbb{Q}_x(X_{T_1} \in dy) = h(x)\frac{\sin\alpha\rho\pi}{\pi}\frac{(1-x)^{\alpha\rho}dy}{(y-x)(y-1)^{\alpha\rho}y^{\alpha(1-\rho)}}.$$

Then it is immediate from Proposition 17 that, with $\tau_x = \inf\{u : \varepsilon(u) > x\}$,

$$\underline{n}(\varepsilon_{\tau_1} \in dy) = \frac{\sin\alpha\rho\pi}{\pi}\frac{dy}{(y-1)^{\alpha\rho}y^{1+\alpha(1-\rho)}}, \text{ and hence}$$

$$\underline{n}(\varepsilon_{\tau_1} < \infty) = \frac{\sin\alpha\rho\pi}{\pi}B(\alpha, 1-\alpha\rho).$$

Since there is no time-dependence, we can argue that

$$\mathbb{P}_0(Y_{T_1} \in dy) = \frac{\underline{n}(\varepsilon_{\tau_1} \in dy)}{\underline{n}(\varepsilon_{\tau_1} < \infty)} = \frac{dy}{B(\alpha, 1-\alpha\rho)(y-1)^{\alpha\rho} y^{1+\alpha(1-\rho)}}.$$

The value of $\mathbb{P}_x(Y_{T_1} \in dy)$ follows by using this in conjunction with (9.7.1) and

$$\mathbb{P}_x(Y_{T_1} \in dy) = \mathbb{P}_x(X_\sigma \in dy) + \mathbb{P}_x(X_\sigma \leq 0)\mathbb{P}_0(Y_{T_1} \in dy) :$$

see Kyprianou [68] for details.

10

Small-Time Behaviour

10.1 Introduction

In this chapter we present some limiting results for a Lévy process as $t \downarrow 0$, being mostly concerned with ideas related to relative stability and attraction to the normal distribution on the one hand and divergence to large values of the Lévy process on the other. These are questions which have been studied in great detail for random walks and in some detail for Lévy processes at ∞, but not so much in the small-time regime. The aim is to find analytical conditions for these kinds of behaviour which are in terms of the *characteristics* of the process, rather than its distribution. Some surprising results occur; for example, we may have $X_t/t \stackrel{P}{\to} +\infty$ $(t \downarrow 0)$ (weak divergence to $+\infty$), whereas $X_t/t \to \infty$ a.s. $(t \downarrow 0)$ is impossible (both are possible when $t \to \infty$), and the former can occur when the negative Lévy spectral component dominates the positive, in a certain sense. "Almost sure stability" of X_t, i.e., X_t/b_t tending to a nonzero constant a.s. as $t \downarrow 0$, where b_t is a non-stochastic measurable function, reduces to the same type of convergence but with normalisation by t, thus is equivalent to "strong law" behaviour. We also consider stability of the overshoot over a one-sided or two-sided barrier, both in the weak and strong sense; in particular we prove the result mentioned in Chapter 6, that in the one-sided case the overshoot is a.s. $o(r)$ as $r \downarrow 0$ if and only if $\delta_+ > 0$.

10.2 Notation and Preliminary Results

Throughout we will make the assumption

$$\Pi(\mathbb{R}) > 0, \tag{10.2.1}$$

since otherwise we are dealing with Brownian motion with drift.

Recall the notations, for $x > 0$,

$$N(x) = \Pi\{(x, \infty)\}, \ M(x) = \Pi\{(-\infty, -x)\}, \tag{10.2.2}$$

the tail sum

$$L(x) = N(x) + M(x), \ x > 0, \tag{10.2.3}$$

and the tail difference

$$D(x) = N(x) - M(x), \ x > 0. \tag{10.2.4}$$

Each of L, N, and M, is non-increasing and right-continuous on $(0, \infty)$ and vanishes at ∞. The rôle of truncated mean is played by

$$A(x) = \gamma + D(1) + \int_1^x D(y)dy, \ x > 0, \tag{10.2.5}$$

and for a kind of truncated second moment we use

$$U(x) = \sigma^2 + 2\int_0^x yL(y)dy. \tag{10.2.6}$$

As previously mentioned, $A(x)$ and $U(x)$ are respectively the mean and variance of $\widetilde{X}_1^x$, where $\widetilde{X}^x$ is the Lévy process we get by replacing each jump in X which is bigger than x, (respectively less than $-x$) by a jump equal to x, (respectively $-x$).

Recall that always $\lim_{x\to 0} U(x) = \sigma^2$ and $\lim_{x\to 0} xA(x) = 0$, and if X is of bounded variation, $\lim_{x\to 0} A(x) = \delta$, the true drift of X.

We start with a few simple, but useful observations.

Lemma 12. *For each $t \geq 0$, $x > 0$, and non-stochastic measurable function $a(t)$*

$$4\mathbb{P}\{|X_t - a(t)| > x\} \geq 1 - e^{-tL(8x)}. \tag{10.2.7}$$

This follows by using symmetrisation and the maximal inequality.

The next result explains why A and U are slowly varying at 0, when the upcoming (10.3.5) or (10.3.23) hold.

Lemma 13. *Let f be any positive differentiable function such that, as $x \uparrow \infty$ ($x \downarrow 0$),*

$$\varepsilon(x) := xf'(x)/f(x) \to 0. \tag{10.2.8}$$

Then f is slowly varying at $\infty(0)$.

Proof. Just note that $f(x) = f(1) \exp \int_1^x y^{-1}\varepsilon(y)dy$ and appeal to the representation theorem for slowly varying functions; see [20], p. 12, Theorem 1.3.1. ■

Finally we note a variant of the Lévy–Itô decomposition, which is proved in exactly the same way that the standard version is:

Lemma 14. *For any fixed $t > 0$ and $1 \geq b > 0$*

$$X_t = A^*(b)t + \sigma B_t + Y^{(1)}_{t,b} + Y^{(2)}_{t,b}, \tag{10.2.9}$$

where

$$A^*(b) = \gamma - \int_{b<|x|<1} x\Pi(dx) = A(b) - bD(b), \tag{10.2.10}$$

$Y^{(1)}_{t,b}$ is the a.s. limit as $\varepsilon \downarrow 0$ of the compensated martingale

$$M^{(1)}_{\varepsilon,t} = \sum_{s\leq t} 1_{\{\varepsilon<|\Delta_s|\leq b\}}\Delta_s - t\int_{\varepsilon<|x|\leq b} x\Pi(dx),$$

$$Y^{(2)}_{t,b} = \sum_{s\leq t:|\Delta_s|>b} \Delta_s,$$

and $B_t, Y^{(1)}_{t,b}$ and $Y^{(2)}_{t,b}$ are independent.

10.3 Convergence in Probability

We start with a "weak law" at 0.

Theorem 34. *There is a non-stochastic δ such that*

$$\frac{X_t}{t} \xrightarrow{P} \delta, \text{ as } t \downarrow 0, \tag{10.3.1}$$

if and only if

$$\sigma^2 = 0, \ \lim_{x\downarrow 0} xL(x) = 0, \text{ and } \lim_{x\downarrow 0} A(x) = \delta. \tag{10.3.2}$$

When (10.3.2) holds, $\int_0^1 D(y)dy$ is conditionally convergent, at least, and satisfies, by (10.2.5),

$$\delta = \gamma + D(1) - \int_0^1 D(y)dy. \tag{10.3.3}$$

This does not imply that X is of bounded variation but if this is true then the δ in (10.3.3) equals the true drift of the process.

The conditions $\lim_{x\to\infty} xL(x) = 0$, and $\lim_{x\to\infty} A(x) = \mu$ are necessary and sufficient for $t^{-1}X_t \xrightarrow{P} \mu$ as $x \to \infty$. So we can think of $A(x)$ as both a generalised mean and a generalised drift.

Proof of Theorem 34. Assume (10.3.2), so $\sigma^2 = 0$, and note that $A^*(t) \to \delta$ as $t \downarrow 0$. Choose $b = t$ in (10.2.9) and note that as $t \downarrow 0$,

$$\begin{aligned}\mathbb{P}\{Y_{t,t}^{(2)} = 0\} &\geq \mathbb{P}\{\text{no jumps with } |\Delta_s| > t \text{ occur by time } t\} \\ &= \exp(-tL(t)) \to 1.\end{aligned}$$

Also $\mathbb{E}(Y_{t,t}^{(1)}) = 0$, and as $t \downarrow 0$,

$$\mathrm{Var}\{t^{-1}Y_{t,t}^{(1)}\} = t^{-1}\int_{|x|<t} x^2 \Pi(dx) \leq t^{-1}\int_0^t 2xL(x)dx \to 0,$$

so $Y_{t,t}^{(1)}/t \xrightarrow{P} 0$ as $t \downarrow 0$, and this establishes (10.3.1) via (10.2.9). On the other hand, if (10.3.1) holds we have $tL(t) \to 0$, by Lemma 12, so we can repeat this argument to see that $t^{-1}\{Y_{t,t}^{(1)} + Y_{t,t}^{(2)}\} \xrightarrow{P} 0$, and from $\sigma t^{-1}B_t + A^*(t) \xrightarrow{P} \delta$ it follows easily that $\sigma = 0$ and $A(t) \to \delta$. ■

Next we look at "relative stability" at 0.

Theorem 35. *There is a non-stochastic measurable function $b(t) > 0$ such that*

$$\frac{X_t}{b(t)} \xrightarrow{P} 1, \text{ as } t \downarrow 0, \tag{10.3.4}$$

if and only if

$$\sigma^2 = 0 \text{ and } \frac{A(x)}{xL(x)} \to \infty, \text{ as } x \downarrow 0. \tag{10.3.5}$$

If these hold, $A(x)$ is slowly varying as $x \downarrow 0$, and $b(t)$ is regularly varying of index 1 as $t \downarrow 0$. Also b may be chosen to be continuous and strictly decreasing to 0 as $t \downarrow 0$, and to satisfy $b(t) = tA(b(t))$ for small enough positive t.

Remark 7. *(We take $\sigma^2 = 0$ throughout this remark). It is possible for (10.3.5) to hold and $\lim_{x\downarrow 0} A(x)$ to be positive, zero, infinite, or non-existent.*

The first of these happens if and only if (10.3.2) holds with $\delta > 0$, so that $X_t/t \xrightarrow{P} \delta > 0$. For the second we require, by (10.3.3), $\gamma + D(1) - \int_0^1 D(y)\,dy = 0$, so that we can then write

$$A(x) = \int_0^x D(y)dy = \int_0^x \{N(y) - M(y)\}dy.$$

Insofar as it implies $A(x) > 0$ for all small enough x, (10.3.5) in this case implies some sort of dominance of the positive Lévy component N over the negative component M. As an extreme case we can have X spectrally positive, i.e. $M(\cdot) \equiv 0$. When this happens N has to be integrable at zero, which implies that X has bounded variation, so in fact a subordinator with drift zero. In these circumstances, (10.3.5) reduces to

$$\frac{xN(x)}{\int_0^x N(y)dy} \to 0 \textit{ as } x \downarrow 0,$$

and from Lemma 13 we see that this happens if and only if $\int_0^x N(y)dy$ *is slowly varying (and tends to 0) at zero.*

The third case can only arise if $\int_x^1 \{M(y) - N(y)\}dy \to \infty$ *as* $x \downarrow 0$, *which implies* $\int_0^1 M(y)dy = \infty$, *so that* X *cannot have bounded variation. This clearly involves some sort of dominance of the negative Lévy component* M *over the positive component* N. *As an extreme case we can have* X *spectrally negative, i.e.* $N(\cdot) \equiv 0$, *so that (10.3.5) becomes*

$$\frac{xL(x)}{A(x)} = \frac{xM(x)}{\gamma - M(1) + \int_x^1 M(y)dy} \sim \frac{xM(x)}{\int_x^1 M(y)dy} \to 0 \text{ as } x \downarrow 0.$$

This happens if and only if $\int_x^1 M(y)dy$ *is slowly varying (and tends to* ∞*) as* $x \downarrow 0$.

Proof of Theorem 35. Assume (10.3.5), and note that condition (10.2.1) implies that $L(t) > 0$ in a neighbourhood of 0, so (10.3.5) implies then that $A(x) > 0$ for all small x, $x \le x_0$, say. A further use of (10.3.5) shows then that for any $z > 0$,

$$\frac{A(x)}{x} \ge zL(x) \tag{10.3.6}$$

for all small enough $x > 0$, and since $L(0+) > 0$ this means that $A(x)/x \to \infty$, as $x \downarrow 0$. Now define $b(t)$ for $t > 0$ by

$$b(t) = \inf\left\{0 < y \le x_0 : \frac{A(y)}{y} \le \frac{1}{t}\right\}. \tag{10.3.7}$$

Then $0 < b(t) < \infty$, $b(t)$ is nondecreasing for $t > 0$, and $b(t) \to 0$ as $t \downarrow 0$. Also, by the continuity of $A(\cdot)$,

$$\frac{tA\left(b(t)\right)}{b(t)} = 1. \tag{10.3.8}$$

This means by (10.3.5) that $tL(b(t)) \to 0$ as $t \to 0$. Next, by Lemma 5.3, $A(\cdot)$ is slowly varying at 0. But (10.3.8) says that $b(\cdot)$ is the inverse of the function $x/A(x)$, and so $b(t)$ is regularly varying with index 1 as $t \downarrow 0$. (See [20], p. 28, Theorem 1.5.12.) It is easy to check by differentiation that $A(x)/x$ strictly increases to ∞ as $x \downarrow 0$, so $b(t)$ is continuous and strictly decreases to 0 as $t \downarrow 0$.

Now we apply (10.2.9) with $b = b(t)$, and $\sigma = 0$. From (10.3.8) and $tL(b(t)) \to 0$ we get $tA^*(b(t))/b(t) \to 1$, and

$$\begin{aligned}\mathbb{P}\{Y^{(2)}_{t,b(t)} = 0\} &\ge \mathbb{P}\{\text{no jumps with } |\Delta| > b(t) \text{ occur by time t}\}\\ &= \exp(-tL(b(t))) \to 1.\end{aligned}$$

Also

$$\operatorname{Var}\left\{Y^{(1)}_{t,b(t)}\right\} = t\int_{|x|<b(t)} x^2 \Pi(dx) = tU(b(t)) + O\{b^2(t)tL(b(t))\}. \quad (10.3.9)$$

By (10.3.5), $xL(x) = o(A(x))$; since A is slowly varying and $\sigma^2 = 0$ it follows that

$$U(x) = 2\int_0^x yL(y)dy = o(xA(x)), \text{ as } x \downarrow 0. \quad (10.3.10)$$

This in turn implies, using (10.3.8), that $tU(b(t)) = o\{b^2(t)\}$ as $t \downarrow 0$. Putting this into (10.3.9) we see that $\operatorname{Var}\{Y^{(1)}_{t,b(t)}/b(t)\} \to 0$, and now (10.2.9) shows that $X_t/b(t) \xrightarrow{P} 1$, i.e. (10.3.4) holds.

For the converse, assume (10.3.4) holds, and note first that this implies that $X^s_t/b(t) \xrightarrow{P} 0$. ($X^s$ is the symmetrised version of X, which has the distribution of $(X - \widetilde{X})/2$, where $\widetilde{X}$ is an independent copy of X.) Then Lemma 12 immediately gives

$$tL(zb(t)) \to 0 \text{ for any fixed } z. \quad (10.3.11)$$

Next, since (10.3.4) implies $-t\Psi(\theta/b(t)) \to i\theta$ for each θ, we have for any fixed $\alpha > 0$

$$\mathbb{E}(\exp\{i\theta X_{\alpha t}/\alpha b(t)\}) = \exp\{-\alpha t\Psi(\theta/\alpha b(t))\} \to \exp(i\theta),$$

on replacing θ by θ/α. This means that $X_{\alpha t}/\alpha b(t) \xrightarrow{P} 1$, so we see easily that $b(\cdot)$ is regularly varying of index 1. Again we use the decomposition (10.2.9) with $b = b(t)$, and as before, get $Y^{(2)}_{t,b(t)}/b(t) \xrightarrow{P} 0$. Thus, with

$$X^*_t = tA^*(b(t)) + \sigma B_t + Y^{(1)}_{t,b(t)},$$

we have $X^*_t/b(t) \xrightarrow{P} 1$. But $\mathbb{E}(\exp\{i\theta X^*_t\}) = \exp\{-t\Psi^*_t(\theta)\}$ where

$$\Psi^*_t(\theta) = -iA^*(b(t))\theta + \frac{1}{2}\sigma^2\theta^2 + \int_{-b(t)}^{b(t)} \left(1 - e^{i\theta x} + i\theta x\right) \Pi(dx).$$

Since the real part of $t\Psi^*_t(\theta/b(t)) \to 0$, we see easily that $\sigma = 0$ and $tU(b(t)) = o\{b^2(t)\}$. Thus $Y^{(1)}_{t,b(t)}/b(t) \xrightarrow{P} 0$, and so we have $tA^*(b(t))/b(t) \to 1$. Combining this with (10.3.11) gives

$$\frac{b(x)L(b(x))}{A^*(b(x))} = \frac{xL(b(x))}{xA^*(b(x))/b(x)} \to 0,$$

so, since $b(\cdot)$ is regularly varying with index 1, $xL(x)/A^*(x) \to 0$. Finally, since

$$\left|\frac{A(x)}{xL(x)} - \frac{A^*(x)}{xL(x)}\right| = \left|\frac{D(x)}{L(x)}\right| \le 1,$$

we see that (10.3.5) holds. ■

Whenever (10.3.4) holds it forces

$$\mathbb{P}\{X_t \ge 0\} \to 1, \text{ as } t \downarrow 0. \tag{10.3.12}$$

We have seen, in Chapter 7, that (10.3.14) below is the necessary and sufficient condition for this, so the result below actually shows that (10.3.12), (10.3.14), and (10.3.13) are equivalent.

Theorem 36. *(i) Suppose* $\sigma^2 > 0$; *then* $\mathbb{P}\{X_t \ge 0\} \to 1/2$ *as* $t \downarrow 0$, *so (10.3.12) implies* $\sigma^2 = 0$.

(ii) Suppose $\sigma^2 = 0$ and $M(0+) > 0$. *There is a non-stochastic measurable function* $b(t) > 0$ *such that*

$$\frac{X_t}{b(t)} \xrightarrow{P} \infty, \text{ as } t \downarrow 0, \tag{10.3.13}$$

whenever

$$\frac{A(x)}{xM(x)} \to \infty, \text{ as } x \downarrow 0, \tag{10.3.14}$$

and this implies (10.3.12). Furthermore, if (10.3.14) holds and $A(x) \to \infty$ *then*

$$\frac{X_t}{t} \xrightarrow{P} \infty, \quad \text{as} \quad t \downarrow 0. \tag{10.3.15}$$

(iii) Suppose X *is spectrally positive, i.e.* $M(x) = 0$ *for all* $x > 0$. *Then (10.3.12) is equivalent to*

$$\sigma^2 = 0 \text{ and } A(x) \ge 0 \text{ for all small } x, \tag{10.3.16}$$

and this happens if and only if X *is a subordinator.*

Remark 8. *Notice that for (10.3.14) to hold and (10.3.5) to fail requires, at least, that* $\limsup_{x\downarrow 0} N(x)/M(x) = \infty$. *It might be thought that this is incompatible with* $\lim_{x\downarrow 0} A(x) = \infty$, *which we have seen entails some kind of dominance of* $M(\cdot)$ *over* $N(\cdot)$. *However the following example satisfies (10.3.14) and has* $\lim_{x\downarrow 0} A(x) = \infty$, *but not (10.3.5), so that* $X_t/t \xrightarrow{P} \infty$, as $t \downarrow 0$, *but there is no* $b(t) > 0$ *with* $X_t/b(t) \xrightarrow{P} 1$.

Example 5. *Take a Lévy process with* $\sigma^2 = 0$, $\gamma = 0$, *and*

$$M(x) = x^{-1} 1_{\{0<x<1\}}, \; N(x) = \frac{c_n}{x_n} 1_{\{x_{n+1} \le x < x_n\}}, \; n \ge 0,$$

where $x_0 = 1$, $x_{n+1} = e^{-\sum_0^n c_r}, n \geq 0$, *the constants* c_n *being defined inductively by* $c_0 = 1$ *and*

$$c_n = \sum_{r=0}^{n-1} c_r e^{-c_r}, \ n \geq 1. \tag{10.3.17}$$

Notice that (10.3.17) implies that $c_n \uparrow \infty$, *and also that*

$$\begin{aligned} A(x_n) &= \int_{x_n}^1 \{M(y) - N(y)\} dy = \log \frac{1}{x_n} - \sum_{r=0}^{n-1} c_r (1 - \frac{x_{r+1}}{x_r}) \\ &= \sum_{r=0}^{n-1} c_r - \sum_{r=0}^{n-1} c_r (1 - e^{-c_r}) = c_n. \end{aligned}$$

It follows that

$$\frac{A(x_n)}{x_n L(x_n)} = \frac{c_n}{1 + c_n} \to 1,$$

so that (10.3.5) fails. Since $xM(x) = 1$, *(10.3.15) is equivalent to* $\lim_{x \downarrow 0} A(x) = \infty$. *Now when* $c_n > 1$, *it is easy to see that* $\inf_{(x_{n+1}, x_n)} A(x) = A(y_n)$, *where* $y_n = x_n / c_n$, *and*

$$\begin{aligned} A(y_n) &= \log \frac{1}{y_n} - \sum_{r=0}^{n-1} c_r (1 - \frac{x_{r+1}}{x_r}) - \frac{c_n}{x_n}(x_n - y_n) \\ &= \sum_{r=0}^{n-1} c_r e^{-c_r} - c_n + 1 + \log c_n = 1 + \log c_n, \end{aligned}$$

and we conclude that (10.3.15) holds.

Remark 9. *In the spectrally positive case (10.3.15) is not possible, because then Theorem 36 guarantees that* X *is a subordinator, in which case* $X_t/t \xrightarrow{P} \delta$ *as* $t \downarrow 0$ *by Theorem 34.*

Proof of Theorem 36. (i) If $\sigma^2 > 0$ it is immediate from (i) of Proposition 4 that

$$\mathbb{E}(\exp i\lambda X_t/\sqrt{t}) = \exp\{-t\Psi(\lambda/\sqrt{t})\} \to \exp(-\sigma^2\lambda^2/2),$$

so that $X_t/\sqrt{t}$ has a limiting $N(0, \sigma^2)$ distribution, as $t \downarrow 0$, and we conclude that $\lim_{t \downarrow 0} \mathbb{P}\{X_t > 0\} = 1/2$.

(ii) This proof is based on a refinement of (10.2.9) with $\sigma = 0$ which takes the form

$$\begin{aligned} X_t = tA(b) &+ \left\{ Y_{t,b}^{(1,+)} + Y_{t,b}^{(2,+)} - tbN(b) \right\} \\ &+ \left\{ Y_{t,b}^{(1,-)} + Y_{t,b}^{(2,-)} + tbM(b) \right\}, \end{aligned} \tag{10.3.18}$$

where $Y_{t,b}^{(1,\pm)}$ and $Y_{t,b}^{(2,\pm)}$ are derived from the positive (respectively, negative) jumps of Δ in the same way that $Y_{t,b}^{(1)}$ and $Y_{t,b}^{(2)}$ are derived from all the jumps of Δ. Since each jump in $Y_{t,b}^{(2,+)}$ is at least b we have the obvious lower bound $Y_{t,b}^{(2,+)} \geq bn^+(t)$, where $n^+(t)$ is the number of jumps in Δ exceeding b which occur by time t.

We start by noting that (10.3.14) and $M(0+) > 0$ imply that $A(x)M(x)/x \to \infty$ as $x \downarrow 0$, so if we put

$$K(x) = \sqrt{xA(x)M(x)},\ x > 0,$$

then also $K(x)/x \to \infty$ as $x \downarrow 0$. As in the previous proof we can therefore define a $b(t) \downarrow 0$ which satisfies, since $K(\cdot)$ is right-continuous,

$$tK(b(t)) = b(t),\ t > 0. \tag{10.3.19}$$

Note that, as $t \downarrow 0$,

$$tM(b(t)) = \frac{b(t)M(b(t))}{K(b(t))} = \sqrt{\frac{b(t)M(b(t))}{A(b(t))}} \to 0. \tag{10.3.20}$$

Using (10.3.18) with $b = b(t)$ we see that $X_t \geq \widetilde{X}_t$ a.s., where

$$\frac{\widetilde{X}_t}{b(t)} = c(t) + \widetilde{Z}_t^+ + Z_t^-,$$

with

$$\widetilde{Z}_t^+ = \frac{Y_{t,b(t)}^{(1,+)}}{b(t)} + n^+(t) - tN(b(t)),$$

$$Z_t^- = \frac{Y_{t,b(t)}^{(1,-)} + Y_{t,b(t)}^{(2,-)}}{b(t)} + tM(b(t)). \tag{10.3.21}$$

By arguments similar to the previous proof we can show that $Z_t^-/c(t) \xrightarrow{P} 0$ and $\widetilde{Z}_t^+/c(t) \xrightarrow{P} 0$ as $t \downarrow 0$, which establishes that $\widetilde{X}_t/(b(t)c(t)) \xrightarrow{\mathbb{P}} 1$ as $t \downarrow 0$. Since $c(t) \to \infty$, we see that $\widetilde{X}_t/b(t) \xrightarrow{P} \infty$ and hence that (10.3.13) holds. Moreover we only need to remark that $b(t)c(t)/t = A(b(t))$ to see that (10.3.15) follows when $A(x) \to \infty$.

(iii) This is proved in [40], but we also proved it in Chapter 7 in a different way. ■

Now we consider attraction of X_t to normality, as $t \downarrow 0$. The original characterisation of $D(N)$ for random walks is in Lévy [73], and $D_0(N)$ is studied in Griffin and Maller [52].

Theorem 37. *$X \in D(N)$, i.e. there are non-stochastic measurable functions $a(t)$, $b(t) > 0$ such that*

$$\frac{X_t - a(t)}{b(t)} \stackrel{D}{\to} N(0,1), \text{ as } t \downarrow 0, \tag{10.3.22}$$

if and only if

$$\frac{U(x)}{x^2 L(x)} \longrightarrow \infty, \quad \text{as } x \downarrow 0. \tag{10.3.23}$$

$X \in D_0(N)$, i.e. there is a non-stochastic measurable function $b(t) > 0$ such that

$$\frac{X_t}{b(t)} \stackrel{D}{\to} N(0,1), \text{ as } t \downarrow 0, \tag{10.3.24}$$

if and only if

$$\frac{U(x)}{x|A(x)| + x^2 L(x)} \to \infty, \text{ as } x \downarrow 0. \tag{10.3.25}$$

If (10.3.23) or (10.3.25) holds, $U(x)$ is slowly varying as $x \downarrow 0$, and $b(t)$ is regularly varying of index $1/2$ as $t\downarrow 0$, and may be chosen to be continuous and strictly decreasing to 0 as $t \downarrow 0$, and to satisfy $b^2(t) = tU(b(t))$ for small enough positive t; furthermore we may take $a(t) = tA(b(t))$ in (10.3.22).

Remark 10. *In the case $L(x) = 0$ for all $x > 0$, X is a Brownian motion and $(X_t - \gamma t)/\sigma\sqrt{t} \stackrel{D}{=} N(0,1)$ for all $t > 0$.*

Remark 11. *The case $b(t) = c\sqrt{t}$, for some $c > 0$, of a square root normalisation, is of special interest in Theorem 37. In this case it is easy to see that (10.3.22) or (10.3.24) holding with $b(t) \sim c\sqrt{t}$ for some $c > 0$ are each equivalent to $\sigma^2 > 0$, and then we may take $c = \sigma$.*

Remark 12. *It is easy to see that, when (10.3.24) holds, the normed process $(X_{t\cdot}/b(t))$ converges in the sense of finite-dimensional distributions to standard Brownian motion. In fact, using Theorem 2.7 of [93], we can conclude that we actually have weak convergence on the space D. A similar comment applies when (10.3.22) holds.*

Proof of Theorem 37. Suppose (10.3.23) holds, so that, by Lemma 13, U is slowly varying. Also since $L(0+) > 0$, $U(x)/x^2 \to \infty$ as $x \downarrow 0$. Hence we can define $b(t) > 0$ by

$$b(t) = \inf\left\{y > 0 : \frac{U(y)}{y^2} \le \frac{1}{t}\right\} \tag{10.3.26}$$

and have $b(t) \downarrow 0$ ($t \downarrow 0$), and

$$tU(b(t)) = b^2(t). \tag{10.3.27}$$

Hence, for all $x > 0$,

$$\frac{tU(xb(t))}{b^2(t)} \to 1 \ (t \downarrow 0), \tag{10.3.28}$$

and then from (10.3.23), for all $x > 0$,

$$tL(xb(t)) \to 0 \ (t \downarrow 0). \tag{10.3.29}$$

Now we apply the decomposition (10.2.9) with $b = b(t)$. In virtue of (10.3.29), we see that $Y^{(2)}_{t,b(t)}/b(t) \stackrel{P}{\to} 0$. Putting $a(t) = tA^*(b(t))$, it suffices to show that if

$$X^{\#}_t = \sigma B_t + Y^{(1)}_{t,b(t)},$$

then $X^{\#}_t/b(t) \stackrel{D}{\to} N(0,1)$. With $\mathbb{E}(\exp\{i\theta X^{\#}_t\}) = \exp\{-t\Psi^{\#}_t(\theta)\}$, this is equivalent to $t\Psi^{\#}_t(\theta/b(t)) \to \theta^2/2$. But

$$\begin{aligned}\Psi^{\#}_t(\theta) &= \frac{1}{2}\sigma^2\theta^2 + \int_{-b(t)}^{b(t)} \left(1 - \mathbb{E}^{i\theta x} - i\theta x\right) \Pi(dx) \\ &= \frac{1}{2}\sigma^2\theta^2 + \int_{-b(t)}^{b(t)} \left\{\frac{1}{2}(\theta x)^2 + o(\theta x)^2\right\} \Pi(dx) \\ &= \frac{\theta^2}{2}\left(\sigma^2 + \left\{\int_{-b(t)}^{b(t)} x^2 \Pi(dx)\right\}\{1 + o(1)\}\right).\end{aligned}$$

Thus

$$t\Psi^{\#}_t(\theta/b(t)) = \frac{t\theta^2}{2b^2(t)}\left\{\sigma^2 + \left\{\int_{-b(t)}^{b(t)} x^2 \Pi(dx)\right\}\{1 + o(1)\}\right\}.$$

Now if $\sigma > 0$ we have $b(t) \sim \sigma\sqrt{t}$, and since the integral tends to 0 we get $t\Psi^{\#}_t(\theta/b(t)) \to \theta^2/2$. If $\sigma = 0$ then we note that

$$\begin{aligned}\frac{t}{b^2(t)}\int_{-b(t)}^{b(t)} x^2 \Pi(dx) &= \frac{t}{b^2(t)}\int_0^{b(t)} x^2 d(-L(x)) \\ &= -tL(b(t)) + \frac{t}{b^2(t)}\int_0^{b(t)} xL(x)dx \\ &= -tL(b(t)) + \frac{tU(b(t))}{b^2(t)} \to 1,\end{aligned}$$

where we have used (10.3.27) and (10.3.29). So again $t\Psi^{\#}_t(\theta/b(t)) \to \theta^2/2$.

The proof of the converse is omitted; see [40]. ■

Next we turn to problems involving overshoots, and begin with weak stability. Define the "two-sided" exit time

$$T(r) = \inf\{t > 0 : |X(t)| > r\}, \quad r > 0. \tag{10.3.30}$$

Theorem 38. *We have*

$$\frac{|X(T(r))|}{r} \xrightarrow{P} 1, \quad \text{as } r \downarrow 0, \tag{10.3.31}$$

if and only if

$$X \in D_0(N) \cup RS \quad (\text{at } 0). \tag{10.3.32}$$

$D_0(N)$ has been defined and characterised in Theorem 37.

RS is the class of processes *relatively stable* at 0; $X \in RS$ if there is a nonstochastic $b(t) > 0$ such that $X(t)/b(t) \xrightarrow{P} \pm 1$ as $t \downarrow 0$. This class has been characterised in Theorem 35.

Proof of Theorem 38. Using the notation and results from Chapter 7, we see easily that there are constants $c_1 > 0, c_2 > 0$ such that for all $\eta > 0, r > 0$,

$$\frac{c_1 L((\eta+1)r)}{k(r)} \leq \mathbb{P}\left\{\frac{|\Delta(T(r))|}{r} > \eta\right\} \leq \frac{c_2 L(\eta r)}{k(r)}, \tag{10.3.33}$$

where we recall

$$k(r) = r^{-1}|A(r)| + r^{-2}U(r).$$

Since $|X(T_r) - r| \leq |\Delta(T(r))|$, we obtain, using (7.3.3), $|X(T(r))|/r \xrightarrow{P} 1$ as $r \downarrow 0$ if and only if

$$\frac{r|A(r)| + U(r)}{r^2 L(r)} \to \infty \text{ as } r \downarrow 0. \tag{10.3.34}$$

The proof is completed by the following result, which is surprising at first sight. However exactly the same result is known in the random-walk case; see Proposition 3.1 of Griffin and McConnell [54], also Lemma 2.1 of Kesten and Maller [63], and Griffin and Maller [52]. Furthermore the proof which is given in Doney and Maller [41] mimics the random-walk proof, so is omitted.

Lemma 15. *In the following, (10.3.34) implies (10.3.35) and (10.3.36), and (10.3.36) implies (10.3.34):*

$$\frac{x|A(x)|}{U(x)} \to 0 \text{ as } x \downarrow 0, \quad \text{or} \quad \liminf_{x\downarrow 0} \frac{x|A(x)|}{U(x)} > 0; \tag{10.3.35}$$

$$\frac{|A(x)|}{xL(x)} \to \infty \text{ as } x \downarrow 0, \quad \text{or} \quad \frac{U(x)}{x|A(x)| + x^2 L(x)} \to \infty \text{ as } x \downarrow 0. \tag{10.3.36}$$

Since (10.3.36) corresponds exactly to $X \in D_0(N) \cup RS$, the result follows. ■

10.4 Almost Sure Results

The following result, which we have already proved, explains why $X_t/t \xrightarrow{\text{a.s.}} \infty$ as $t \downarrow 0$ cannot occur.

Theorem 39. *If X has bounded variation then:*

$$\lim_{t\downarrow 0} \frac{X_t}{t} = \delta \text{ a.s.,} \tag{10.4.1}$$

where δ is the drift; if X has infinite variation then

$$-\infty = \liminf_{t\downarrow 0} \frac{X_t}{t} < \limsup_{t\downarrow 0} \frac{X_t}{t} = +\infty \text{ a.s.} \tag{10.4.2}$$

Now we turn to a.s. relative stability.

Theorem 40. *There is a non-stochastic measurable function $b(t) > 0$ such that*

$$\frac{X_t}{b(t)} \overset{\text{a.s.}}{\to} 1, \text{as } t \downarrow 0, \tag{10.4.3}$$

if and only if the drift coefficient δ is well defined, $\delta > 0$, and

$$\frac{X_t}{t} \overset{\text{a.s.}}{\to} \delta, \text{as } t \downarrow 0. \tag{10.4.4}$$

Proof of Theorem 40. Let (10.4.3) hold and we will prove (10.4.4). Define

$$W_j = X(2^{-j}) - X(2^{-j-1}),$$

which are independent rvs with the same distribution as $X(2^{-j} - 2^{-j-1}) = X(2^{-j-1})$. Also

$$\frac{X(2^{-n})}{b(2^{-n})} = \frac{1}{b(2^{-n})} \sum_{j=n}^{\infty} \left(X(2^{-j}) - X(2^{-j-1})\right) = \frac{1}{b(2^{-n})} \sum_{j=n}^{\infty} W_j.$$

By (10.4.3), $\limsup_{n\to\infty} |X(2^{-n})|/b(2^{-n}) < \infty$ a.s, hence

$$\limsup_{n\to\infty} \frac{|W_n|}{b(2^{-n})} = \limsup_{n\to\infty} \frac{|\sum_{j=n}^{\infty} W_j - \sum_{j=n+1}^{\infty} W_j|}{b(2^{-n})} < \infty \text{ a.s.}$$

The W_j are independent, so by the Borel–Cantelli lemma, $\sum_{n\geq 0} \mathbb{P}\{|W_n| > cb(2^{-n})\}$ converges for some $c > 0$.

Now X_t is weakly relatively stable at 0 so we know from Theorem 35 that $\sigma^2 = 0$, $A(x) > 0$ for all small x, and $b(t)$ can be taken to be continuous, strictly increasing, regularly varying with index 1 as $t \downarrow 0$, and to satisfy $b(t) = tA(b(t))$ for all small $t > 0$. Note that we can write $W_n = W_{n+1} + W'_{n+1}$, where W'_{n+1} is an independent copy of W_{n+1}, so that

$$\begin{aligned}\mathbb{P}\left\{|W_n| > \frac{c}{2}b(2^{-n})\right\} &= \mathbb{P}\left\{|W_{n+1} + W'_{n+1}| > \frac{c}{2}b(2^{-n})\right\} \\ &\leq 2\mathbb{P}\left\{|W_{n+1}| > \frac{c}{2}b(2^{-n})\right\}.\end{aligned}$$

Using this and the regular variation of $b(\cdot)$ we see that $\sum \mathbb{P}\{|W_n| > cb(2^{-n})\}$ converges for all $c > 0$. We also get from the proof of Theorem 35 that $xL(b(x)) \to 0$ as $x \downarrow 0$, so by Lemma 12,

$$\mathbb{P}\{|X_t| > b(t)\} \geq ctL(b(t))$$

for all small t, for some $c > 0$. Thus

$$\infty > \sum_{n \geq 0} \mathbb{P}\left\{|X(2^{-n-1})| > b(2^{-n})\right\} \geq c \sum_n 2^{-n} L(b(2^{-n})),$$

from which we see that

$$\int_0^1 L(b(x))dx < \infty. \tag{10.4.5}$$

From this we can deduce that

$$\int_0^1 \frac{L(x)}{A(x)} dx < \infty, \tag{10.4.6}$$

and then that $\int_0^1 L(x)dx < \infty$. Thus we can define the drift coefficient δ and write

$$A(x) = \gamma + D(1) - \int_x^1 D(y)dy = \delta + \int_0^x D(y)dy.$$

Now $A(x) > 0$ near 0, so we must have $\delta \geq 0$. If $\delta = 0$ we have

$$|A(x)| = |\int_0^x D(y)dy| \leq \int_0^x L(y)dy,$$

so from (10.4.6)

$$\int_0^b \frac{L(y)dy}{\int_0^y L(x)dx} < \infty$$

which is impossible. It follows that $\delta > 0$. Then $A(x) \to \delta$ as $x \downarrow 0$ so $b(t) \sim t\delta$ as $t \downarrow 0$, and (10.4.4) follows from (10.4.3). ■

The next theorem characterises a.s. stability of the overshoot in the two-sided case.

Theorem 41. *We have*

$$\frac{|X(T(r))|}{r} \overset{a.s.}{\to} 1 \quad as \quad r \downarrow 0, \tag{10.4.7}$$

if and only if

$$\sigma^2 > 0 \text{ or } \sigma^2 = 0, \text{ and } X \text{ has bounded variation and drift } \delta \neq 0. \tag{10.4.8}$$

Remark 13. *It should be noted that the two situations in which (10.4.8) hold are completely different. In the first case the probability that* X *exits the interval at the top tends to* $1/2$*, whereas in the second case* $X(T(r))/r \overset{a.s.}{\to} 1$ *if* $\delta > 0$ *and* $X(T(r))/r \overset{a.s.}{\to} -1$ *if* $\delta < 0$.

Proof of Theorem 41. The crux of the matter is that, using the bound (10.3.33), it is possible to show that (10.4.7) occurs if and only if

$$\int_0^1 \frac{xL(x)dx}{x|A(x)| + U(x)} < \infty. \tag{10.4.9}$$

To see this choose $0 < \lambda < 1$ and $0 < \varepsilon < 1$ and assume (10.4.9). By (10.3.33), for some $c > 0$,

$$\sum_{n \geq 0} \mathbb{P}\{|\Delta(T(\lambda^n))| > \varepsilon\lambda^n\} \leq c \sum_{n \geq 0} \frac{L(\varepsilon\lambda^n)}{k(\varepsilon\lambda^n)}$$

$$\leq c \sum_{n \geq 0} \frac{\lambda^{-n}}{1-\lambda} \int_{\lambda^{n+1}}^{\lambda^n} \frac{L(\varepsilon y)dy}{k(\varepsilon\lambda^n)} \leq \frac{3c\lambda^{-3}}{1-\lambda} \sum_{n \geq 0} \int_{\lambda^{n+1}}^{\lambda^n} \frac{L(\varepsilon y)}{yk(\varepsilon y)} dy$$

$$= \frac{3c\lambda^{-3}}{1-\lambda} \int_0^1 \frac{L(\varepsilon y)dy}{yk(\varepsilon y)} = \frac{3c\lambda^{-3}}{1-\lambda} \int_0^\varepsilon \frac{L(y)dy}{yk(y)} < \infty.$$

Thus

$$\frac{\Delta(T(\lambda^n))}{\lambda^n} \to 0 \text{ a.s., as } n \to \infty, \tag{10.4.10}$$

and (10.4.7) follows.

Conversely, let (10.4.7) hold. Then, as $n \to \infty$,

$$\frac{|\Delta(T(2^{-n}))|}{2^{-n}} = \frac{|X(T(2^{-n})) - X(T(2^{-n})-)|}{2^{-n}} \overset{a.s.}{\leq} (1+\varepsilon) + 1 = 2 + \varepsilon.$$

Thus if we write $B_n = \{|\Delta(T(2^{-n}))| > (2+\varepsilon)2^{-n}\}$, we have

$$\mathbb{P}\{B_n \text{ i.o.}\} = 0. \tag{10.4.11}$$

Suppose we have

$$\sum \mathbb{P}\{B_n\} < \infty. \tag{10.4.12}$$

Then we easily get

$$\int_0^1 \frac{xL(x)dx}{x|A(x)| + U(x)} < \infty.$$

So we need to deduce (10.4.12) from (10.4.11). This can be done using a version of the Borel–Cantelli lemma, modifying the working of Griffin and

Maller, [52]. The hard part, which we omit, is to show that, were (10.4.12) false, we would have

$$\sum_{m=1}^{n-1}\sum_{l=m+1}^{n}\mathbb{P}(B_m\cap B_l)\le (c_1+o(1))\left(\sum_{m=1}^{n}\mathbb{P}(B_m)\right)^2. \tag{10.4.13}$$

By Spitzer ([94], p. 317) this implies $\mathbb{P}(B_n \text{ i.o.}) > 0$, which is impossible if (10.4.11) holds. Hence $\sum_{n\ge 1}\mathbb{P}(B_n) < \infty$ and we have (10.4.12). To finish the proof we need the following analytic fact: (10.4.9) occurs if and only if (10.4.8) occurs. We will take this for granted, as it's proof again follows closely the random-walk argument. ■

Now define the "one-sided" exit time

$$T^*(r) = \inf\{t > 0 : X(t) > r\}, \quad r > 0. \tag{10.4.14}$$

As usual, let H_+ be the upwards ladder height subordinator associated with X, and let δ_+ be its drift. Define

$$T_+^*(r) = \inf\{t > 0 : H_+(t) > r\},\ r > 0. \tag{10.4.15}$$

Then clearly

$$X(T^*(r)) = H_+(T_+^*(r)). \tag{10.4.16}$$

Theorem 42. *We have*

$$\frac{X(T^*(r))}{r} \stackrel{\text{a.s.}}{\to} 1 \quad\text{as}\quad r\downarrow 0, \tag{10.4.17}$$

if and only if $\delta_+ > 0$.

Proof of Theorem 42. Simply use (10.4.16) to see that (10.4.17) is equivalent to $H_+(T_+^*(r))/r \to 1$ a.s. Of course $T_+^*(r)$ is also the two-sided exit time for H_+, so (10.4.8) holds for H_+, and conversely. This is only possible if $\delta_+ > 0$. ■

Remark 14. *A similar argument using Theorem 38 shows that weak relative stability of the one-sided overshoot is equivalent to $H_+ \in RS$, and according to Remark 7 this happens if and only if*

$$\overline{\mu}_+(x) \text{ is slowly varying as } x\downarrow 0.$$

What is this equivalent to for the characteristics of X? Note that this certainly happens when $\delta_+ > 0$, so the solution to this problem would provide an interesting extension of Vigon's result Theorem 20, which characterizes all Lévy processes having $\delta_+ > 0$.

10.5 Summary of Asymptotic Results

Recall the notations, for $x > 0$,

$$N(x) = \Pi\{(x, \infty)\},\ M(x) = \Pi\{(-\infty, -x)\},$$
$$L(x) = N(x) + M(x),\ D(x) = N(x) - M(x),$$
$$A(x) = \gamma + D(1) + \int_1^x D(y)dy,\ \text{and}$$
$$U(x) = \sigma^2 + 2\int_0^x yL(y)dy.$$

Also $X \in D(N)$ means $\exists\ a(t)$ and $b(t) > 0$ such that $\frac{X_t - a(t)}{b(t)} \xrightarrow{D} N(0,1)$, $X \in D_0(N)$ means this is possible with $a(t) \equiv 0$, and $X \in RS$ means $\exists$ $b(t) > 0$ such that $\frac{X_t}{b(t)} \xrightarrow{P} 1$ or -1.

10.5.1 Laws of Large Numbers

The small-time results, assuming $\sigma^2 = 0$, are.

(i) $t^{-1}X_t \xrightarrow{P} \delta \in \mathbb{R} \Longleftrightarrow xL(x) \to 0,\ A(x) \to \delta$.

(ii) $t^{-1}X_t \xrightarrow{a.s.} \delta \in \mathbb{R} \Longleftrightarrow X$ has bounded variation, δ is the drift.

(iii) $\exists b > 0$ with $b(t)^{-1}X_t \xrightarrow{P} 1 \Longleftrightarrow A(x)/xL(x) \to \infty$, and $X \in RS \Longleftrightarrow |A(x)|/xL(x) \to \infty$.

(iv) $\exists b > 0$ with $b(t)^{-1}X_t \xrightarrow{a.s.} 1 \Longleftrightarrow X$ has bounded variation and drift $\delta > 0$, $b(t) \backsim \delta t$.

(v) $\exists b > 0$ with $b(t)^{-1}X_t \xrightarrow{P} \infty \Longleftrightarrow \mathbb{P}(X_t > 0) \to 1 \Longleftrightarrow A(x)/xM(x) \to \infty$.

(And we can take $b(t) = t$ if also $A(x) \to \infty$.)

(vi) $t^{-1}X_t \xrightarrow{a.s.} \infty$ is not possible.

The corresponding large-time results are similar, except we can allow $\sigma^2 > 0$. In (i), (ii), and (iv), δ is replaced by $\mu = \mathbb{E}X_1$. In (v) we can add $X_t \xrightarrow{P} \infty$ to the equivalences, but (vi) is different. $t^{-1}X_t \xrightarrow{a.s.} \infty$ as $t \to \infty$ is possible, and it is equivalent to $X_t \xrightarrow{a.s.} \infty$ as $t \to \infty$. The NASC for this is given in Erickson's test, at the end of Chapter 6.

10.5.2 Central Limit Theorems

The small-time results, assuming $\sigma^2 = 0$, are.

(i) $X \in D(N) \Longleftrightarrow U(x)/x^2L(x) \to \infty$.

(ii) $X \in D_0(N) \Longleftrightarrow U(x)/(x^2L(x) + x|A(x)|) \to \infty$.

The large-time results are the same, except that we can allow $\sigma^2 > 0$.

10.5.3 Exit from a Symmetric Interval

Here $T_r = \inf\{t : |X_t| > r\}$ denotes the exit time and $O_r = X_{T_r} - r$ the corresponding overshoot. The small-time results are.

(i) $\mathbb{P}(O_r > 0) \to 1 \Longleftrightarrow \mathbb{P}(X_t > 0) \to 1 \Longleftrightarrow A(x)/xM(x) \to \infty$.

(ii) $r^{-1}O_r \stackrel{P}{\to} 0 \Longleftrightarrow X \in RS \cup D_0(N)$.

(iii) $r^{-1}O_r \stackrel{a.s.}{\to} 0 \Longleftrightarrow \sigma^2 > 0$ or $\sigma^2 = 0$, X has bounded variation, and $\delta \neq 0$.

The large-time results are similar, except that in (iii) the condition is that either $\mathbb{E}X_1^2 < \infty$ and $\mathbb{E}X_1 = 0$, or $\mathbb{E}|X_1| < \infty$ and $\mathbb{E}X_1 \neq 0$.

Acknowledgement

I would like to thank the organisers, and in particular Jean Picard, for offering me the opportunity of delivering these lectures at St Flour, and the participants for their comments and suggestions. I would also like to thank my colleagues and co-authors whose work appears here, including Larbi Alili, Peter Andrew, Jean Bertoin, Loic Chaumont, Cindy Greenwood, Andreas Kyprianou, Ross Maller, Philippe Marchal, Vincent Vigon and Matthias Winkel. Finally my thanks are due to my students, Elinor Jones and Mladen Savov, for their careful reading of the manuscript.

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